

Contents

<i>Introduction</i>	<i>page 1</i>
1 Semigroups and Generators	9
1.1 Motivation from Partial Differential Equations	9
1.2 Definition of a Semigroup and Examples	10
1.3 Unbounded Operators and Generators	15
1.3.1 Unbounded Operators and Density of Generators	15
1.3.2 Differential Equations in Banach Space	18
1.3.3 Generators as Closed Operators	20
1.3.4 Closures and Cores	21
1.4 Norm-Continuous Semigroups	23
1.5 The Resolvent of a Semigroup	25
1.5.1 The Resolvent of a Closed Operator	25
1.5.2 Properties of the Resolvent of a Semigroup	27
1.6 Exercises for Chapter 1	29
2 The Generation of Semigroups	31
2.1 Yosida Approximants	31
2.2 Classifying Generators	33
2.3 Applications to Parabolic PDEs	38
2.3.1 Bilinear Forms, Weak Solutions and the Lax–Milgram Theorem	39
2.3.2 Energy Estimates and Weak Solutions to the Elliptic Problem	40
2.3.3 Semigroup Solution of the Parabolic Problem	41
2.4 Exercises for Chapter 2	43
3 Convolution Semigroups of Measures	46
3.1 Heat Kernels, Poisson Kernels, Processes and Fourier Transforms	46

3.1.1	The Gauss–Weierstrass Function and the Heat Equation	46
3.1.2	Brownian Motion and Itô’s Formula	50
3.1.3	The Cauchy Distribution, the Poisson Kernel and Laplace’s Equation	52
3.2	Convolution of Measures and Weak Convergence	55
3.2.1	Convolution of Measures	55
3.2.2	Weak Convergence	57
3.3	Convolution Semigroups of Probability Measures	59
3.4	The Lévy–Khintchine Formula	64
3.4.1	Stable Semigroups	69
3.4.2	Lévy Processes	71
3.5	Generators of Convolution Semigroups	73
3.5.1	Lévy Generators as Pseudo-Differential Operators	76
3.6	Extension to L^p	78
3.7	Exercises for Chapter 3	81
4	Self-Adjoint Semigroups and Unitary Groups	83
4.1	Adjoint Semigroups and Self-Adjointness	83
4.1.1	Positive Self-Adjoint Operators	85
4.1.2	Adjoints of Semigroups on Banach Spaces	88
4.2	Self-Adjointness and Convolution Semigroups	89
4.3	Unitary Groups, Stone’s Theorem	91
4.4	Quantum Dynamical Semigroups	97
4.5	Exercises for Chapter 4	100
5	Compact and Trace Class Semigroups	102
5.1	Compact Semigroups	102
5.2	Trace Class Semigroups	104
5.2.1	Hilbert–Schmidt and Trace Class Operators	104
5.2.2	Trace Class Semigroups	108
5.2.3	Convolution Semigroups on the Circle	109
5.2.4	Quantum Theory Revisited	112
5.3	Exercises for Chapter 5	114
6	Perturbation Theory	116
6.1	Relatively Bounded and Bounded Perturbations	116
6.1.1	Contraction Semigroups	116
6.1.2	Analytic Semigroups	119
6.2	The Lie–Kato–Trotter Product Formula	120
6.3	The Feynman–Kac Formula	123

6.3.1	The Feynman–Kac Formula via the Lie–Kato–Trotter Product Formula	123
6.3.2	The Feynman–Kac Formula via Itô’s Formula	125
6.4	Exercises for Chapter 6	127
7	Markov and Feller Semigroups	129
7.1	Definitions of Markov and Feller Semigroups	129
7.2	The Positive Maximum Principle	133
7.2.1	The Positive Maximum Principle and the Hille–Yosida–Ray Theorem	133
7.2.2	Crash Course on Distributions	135
7.2.3	The Courrège Theorem	136
7.3	The Martingale Problem	144
7.3.1	Sub-Feller Semigroups	147
8	Semigroups and Dynamics	149
8.1	Invariant Measures and Entropy	149
8.1.1	Invariant Measures	149
8.1.2	Entropy	151
8.2	Semidynamical Systems	152
8.2.1	Koopmanism	152
8.2.2	Dynamical Systems and Differential Equations	157
8.3	Approaches to Irreversibility	161
8.3.1	Dilations of Semigroups	162
8.3.2	The Misra–Prigogine–Courbage Approach	164
9	Varopoulos Semigroups	166
9.1	Prelude – Fractional Calculus	166
9.2	The Hardy–Littlewood–Sobolev Inequality and Riesz Potential Operators	168
9.3	Varopoulos Semigroups	169
9.4	Varopoulos’s Theorem	173
9.5	Nash Inequality, Symmetric Diffusion Semigroups and Heat Kernels	175
	<i>Notes and Further Reading</i>	179
Appendix A	The Space $C_0(\mathbb{R}^d)$	182
Appendix B	The Fourier Transform	186
Appendix C	Sobolev Spaces	190

<i>Appendix D</i>	Probability Measures and Kolmogorov's Theorem on Construction of Stochastic Processes	194
<i>Appendix E</i>	Absolute Continuity, Conditional Expectation and Martingales	197
	E.1 The Radon–Nikodym Theorem	197
	E.2 Conditional Expectation	200
	E.3 Martingales	203
<i>Appendix F</i>	Stochastic Integration and Itô's Formula	205
	F.1 Stochastic Integrals	205
	F.2 Itô's Formula	208
<i>Appendix G</i>	Measures on Locally Compact Spaces – Some Brief Remarks	212
	<i>References</i>	214
	<i>Index</i>	219