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978-1-108-48229-5 — An Invitation to Applied Category Theory
Brendan Fong , David I. Spivak
Frontmatter
More Information

An Invitation to Applied Category Theory

Category theory is unmatched in its ability to organize and layer abstractions and to find commonalities between structures of all sorts. No longer the exclusive preserve of pure mathematicians, it is now proving itself to be a powerful tool in science, informatics, and industry. By facilitating communication between communities and building rigorous bridges between disparate worlds, applied category theory has the potential to be a major organizing force.

This book offers a self-contained tour of applied category theory. Each chapter follows a single thread motivated by a real-world application and discussed with category-theoretic tools. We see data migration as an adjoint functor, electrical circuits in terms of monoidal categories and operads, and collaborative design via enriched profunctors. All the relevant category theory, from simple to sophisticated, is introduced in an accessible way with many examples and exercises, making this an ideal guide even for those without experience of university-level mathematics.

Brendan Fong is a postdoctoral associate in the Department of Mathematics at the Massachusetts Institute of Technology. His research explores how we use pictures to represent and reason about the systems around us, and how to understand the world from a relational point of view. These topics find their intersection in applied category theory.

David I. Spivak is a research scientist in the Department of Mathematics at the Massachusetts Institute of Technology. He has found applications of category theory ranging from database integration to knowledge representation, from materials science to dynamical systems and behavior. He is the author of two other books in category theory.

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Seven Sketches in Compositionality

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More Information

To our parents

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978-1-108-48229-5 — An Invitation to Applied Category Theory
Brendan Fong , David I. Spivak
Frontmatter
More Information

Contents

Preface *page* ix

1 Generative Effects: Orders and Galois Connections 1

1.1 More Than the Sum of Their Parts 1
1.2 What is Order? 7
1.3 Meets and Joins 23
1.4 Galois Connections 27
1.5 Summary and Further Reading 36

2 Resource Theories: Monoidal Preorders and Enrichment 38

2.1 Getting from a to b 38
2.2 Symmetric Monoidal Preorders 40
2.3 Enrichment 56
2.4 Constructions on $\mathcal{V}$-Categories 63
2.5 Computing Presented $\mathcal{V}$-Categories with Matrix Multiplication 68
2.6 Summary and Further Reading 75

3 Databases: Categories, Functors, and Universal Constructions 77

3.1 What is a Database? 77
3.2 Categories 81
3.3 Functors, Natural Transformations, and Databases 89
3.4 Adjunctions and Data Migration 101
3.5 Bonus: An Introduction to Limits and Colimits 108
3.6 Summary and Further Reading 115

4 Collaborative Design: Profunctors, Categorification, and Monoidal Categories 117

4.1 Can We Build It? 117
4.2 Enriched Profunctors 119
4.3 Categories of Profunctors 125
4.4 Categorification 133
4.5 Profunctors Form a Compact Closed Category 140
4.6 Summary and Further Reading 145

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CAMBRIDGE

Cambridge University Press
978-1-108-48229-5 — An Invitation to Applied Category Theory
Brendan Fong , David I. Spivak
Frontmatter
More Information

5 Signal Flow Graphs: Props, Presentations, and Proofs 147

5.1 Comparing Systems as Interacting Signal Processors 147
5.2 Props and Presentations 149
5.3 Simplified Signal Flow Graphs 159
5.4 Graphical Linear Algebra 168
5.5 Summary and Further Reading 179

6 Electric Circuits: Hypergraph Categories and Operads 180

6.1 The Ubiquity of Network Languages 180
6.2 Colimits and Connection 183
6.3 Hypergraph Categories 196
6.4 Decorated Cospans 202
6.5 Operads and Their Algebras 210
6.6 Summary and Further Reading 218

7 Logic of Behavior: Sheaves, Toposes, and Internal Languages 219

7.1 How Can We Prove Our Machine is Safe? 219
7.2 The Category Set as an Exemplar Topos 222
7.3 Sheaves 230
7.4 Toposes 241
7.5 A Topos of Behavior Types 250
7.6 Summary and Further Reading 255

Appendix: Exercise Solutions 256

A.1 Solutions for Chapter 1 256
A.2 Solutions for Chapter 2 268
A.3 Solutions for Chapter 3 275
A.4 Solutions for Chapter 4 285
A.5 Solutions for Chapter 5 292
A.6 Solutions for Chapter 6 303
A.7 Solutions for Chapter 7 313

References 325
Index 331

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Cambridge University Press
978-1-108-48229-5 — An Invitation to Applied Category Theory
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Frontmatter
More Information

Preface

Category theory is becoming a central hub for all of pure mathematics. It is unmatched in its ability to organize and layer abstractions, to find commonalities between structures of all sorts, and to facilitate communication between different mathematical communities.

But it has also been branching out into science, informatics, and industry. We believe that it has the potential to be a major cohesive force in the world, building rigorous bridges between disparate worlds, both theoretical and practical. The motto at MIT is *mens et manus*, Latin for mind and hand. We believe that category theory – and pure math in general – has stayed in the realm of mind for too long; it is ripe to be brought to hand.

Purpose and audience

The purpose of this book is to offer a self-contained tour of applied category theory. It is an invitation to discover advanced topics in category theory through concrete real-world examples. Rather than try to give a comprehensive treatment of these topics – which include adjoint functors, enriched categories, proarrow equipments, toposes, and much more – we merely provide a taste of each. We want to give readers some insight into how it feels to work with these structures as well as some ideas about how they might show up in practice.

The audience for this book is quite diverse: anyone who finds the above description intriguing. This could include a motivated high school student who hasn't seen calculus yet but has loved reading a weird book on mathematical logic they found at the library. Or a machine-learning researcher who wants to understand what vector spaces, design theory, and dynamical systems could possibly have in common. Or a pure mathematician who wants to imagine what sorts of applications their work might have. Or a recently retired programmer who's always had an eerie feeling that category theory is what they've been looking for to tie it all together, but who's found the usual books on the subject impenetrable.

For example, we find it something of a travesty that at the time of publication there is almost no introductory material available on monoidal categories. Even beautiful modern introductions to category theory, e.g. by Riehl [Rie17] or Leinster [Lei14], do not include anything on this rather central topic. The only exceptions we can think of are [CK17, Chapter 3] and [CP10], each of which has a very user-friendly introduction to monoidal categories; however, readers who are not drawn to physics may not think to look there.

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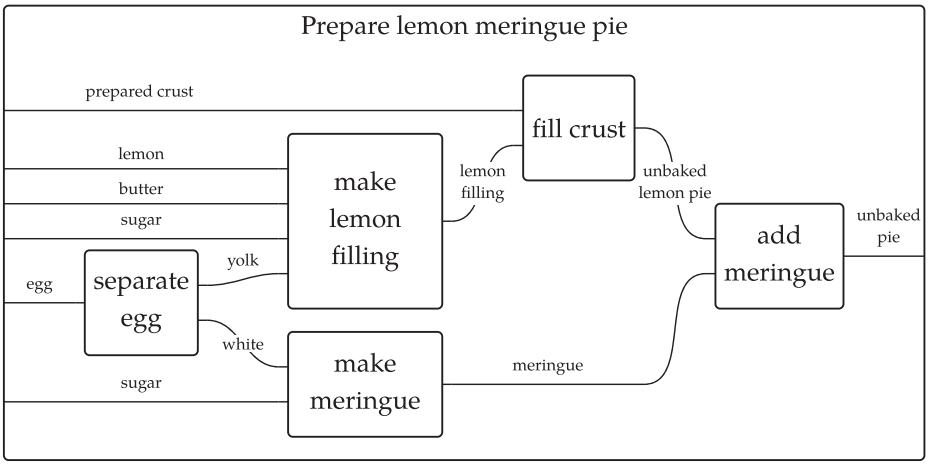
Frontmatter
More Information

The basic idea of monoidal categories is certainly not too abstract; modern human intuition seems to include a pre-theoretical understanding of monoidal categories that is just waiting to be formalized. Is there anyone who wouldn't correctly understand the basic idea being communicated in the following diagram?



Many applied category theory topics seem to take monoidal categories as their jumping-off point. So one aim of this book is to provide a reference – even if unconventional – for this important topic.

We hope this book inspires both new visions and new questions. We intend it to be self-contained in the sense that it is approachable with minimal prerequisites, but not in the sense that the complete story is told here. On the contrary, we hope that readers use this as an invitation to further reading, to orient themselves in what is becoming a large literature, and to discover new applications for themselves.

This book is, unashamedly, our take on the subject. While the abstract structures we explore are important to any category theorist, the specific topics have simply been chosen to our personal taste. Our examples are ones that we find simple but powerful, concrete but representative, entertaining but in a way that feels important and expansive at the same time. We hope our readers will enjoy themselves and learn a lot in the process.

How to read this book

The basic idea of category theory – which threads through every chapter – is that if one pays careful attention to structures and coherence, the resulting systems will be extremely reliable and interoperable. For example, a category involves several structures: a collection of objects, a collection of morphisms relating objects, and a formula for combining any chain of morphisms into a morphism. But these structures need to *cohere* or work together in a simple commonsense way: a chain of chains is itself a long

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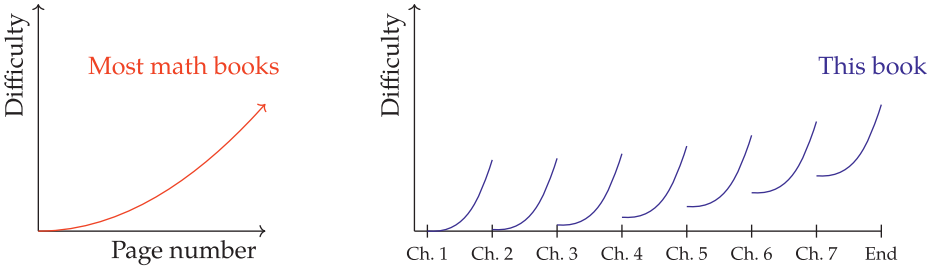
Frontmatter
More Information

chain, so combining a chain of chains should be the same as combining the long chain. That's it!

We shall see structures and coherence come up in pretty much every definition we give: "here are some things and here are how they fit together." We ask the reader to be on the lookout for structures and coherence as they read the book, and to realize that as we layer abstraction upon abstraction, it is the coherence that makes all the parts work together harmoniously in concert.

Each chapter in this book is motivated by a real-world topic, such as electrical circuits, control theory, cascade failures, information integration, and hybrid systems. These motivations lead us into and through various sorts of category-theoretic concepts. We generally have one motivating idea and one category-theoretic purpose per chapter, and this forms the title of the chapter, e.g. Chapter 4 is "Co-design: Profunctors, Categorification, and Monoidal Categories."

In many math books, the difficulty is roughly a monotonically increasing function of the page number. In this book, this occurs in each chapter, but not so much in the book as a whole. The chapters start out fairly easy and progress in difficulty.



The upshot is that if you find the end of a chapter very difficult, hope is certainly not lost: you can start on the next one and make good progress. This format lends itself to giving you a first taste now, but also leaving open the opportunity for you to come back to the book at a later date and get more deeply into it. But by all means, if you have the gumption to work through each chapter to its end, we very much encourage that!

We include about 240 exercises throughout the text, with solutions in the Appendix at the end of the book. Usually these exercises are fairly straightforward; the only thing they demand is that readers change their mental state from passive to active, reread the previous paragraphs with intent, and put the pieces together. Readers become *students* when they work through the exercises; until then they are more tourists, riding on a bus and listening off and on to the tour guide. Hey, there's nothing wrong with that, but we do encourage you to get off the bus and make direct contact with the native population and local architecture as often as you can.

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Frontmatter
More Information

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We also thank our sponsors at the AFOSR; this work was supported by grants FA9550–14–1–0031 and FA9550–17–1–0058.

Finally, we extend a very special thanks to John Baez for running an online course (https://forum.azimuthproject.org) on this material and generating tons of great feedback.

Personal note

Our motivations to apply category theory outside of math are, perhaps naively, grounded in the hope it can help bring humanity together to solve our big problems. But category theory is a tool for thinking, and like any tool it can be used for purposes we align with and those we don't.

In this personal note, we ask that readers try to use what they learn in this book to do something they would call "good," in terms of contributing to the society they'd want to live in. For example, if you're planning to study this material with others, consider specifically inviting someone from an underrepresented minority – a group that is more highly represented in society than in upper-level math classes – to your study group. As another example, perhaps you can use the material in this book to design software that helps people relate to and align with each other. What is the mathematics of a well-functioning society?

The way we use our tools affects all our lives. Our society has seen the results – both the wonders and the waste – resulting from rampant selfishness. We would be honored if readers found ways to use category theory as part of an effort to connect people, to create common ground, to explore the cross-cutting categories in which life, society, and environment can be represented, and to end the ignorance entailed by limiting ourselves to a singular ontological perspective on anything.

If you do something of the sort, please let us and the community know about it.

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