

Quantum Field Theory

This modern text combines fundamental principles with advanced topics and recent techniques in a rigorous and self-contained treatment of quantum field theory.

Key features include:

- a review of basic principles, starting with quantum mechanics and special relativity, and covering elementary aspects of quantum field theory and perturbation theory;
- standard results and tools relevant to many applications, including canonical quantization, path integrals, non-Abelian gauge theories and the renormalization group;
- advanced topics such as effective field theories, quantum anomalies, stable extended field configurations, lattice field theory and field theory at a finite temperature or in the strong field regime;
- two chapters dedicated to new methods for calculating scattering amplitudes (spinor-helicity, on-shell recursion and generalized unitarity), equipping students with practical skills for future research; and
- an accessible style featuring numerous worked examples, applications and end-of-chapter problems.

This is an essential text for graduate students in physics, and an equally excellent reference for researchers in the field.

Quantum Field Theory

From Basics to Modern Topics

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To Kanako and Nathan.

In order to improve the mind,
we ought less to learn,
than to contemplate.

RENÉ DESCARTES

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Preface

This book started in the form of lecture notes for a quantum field theory course I taught at the Master's program on High Energy Physics at École Polytechnique (Palaiseau, France). Students enrolled in these courses have already had an introductory course to quantum field theory, and therefore I started directly with more advanced topics, such as path integral quantization, non-Abelian gauge theories and the renormalization group.

By virtue of the audience to which these lectures were delivered, a large portion of the material in these notes was directed toward applications in particle physics. However, in the last 25 years the traditional workflow (Lagrangian \rightarrow Feynman rules \rightarrow scattering amplitudes \rightarrow cross-sections) has been shaken by a flood of new developments that has shattered how we view and calculate scattering amplitudes. More importantly, these new methods uncovered the great unity that exists across seemingly unrelated quantum field theories, something that was impossible to foresee from their respective Feynman rules. Thus, one of the goals I set for these lecture notes was to include – in addition to the traditional methods, not as a replacement of them – a detailed exposition of these new techniques, which are not addressed in most of the existing textbooks because they did not exist or were not mature enough when they were written.

At the same time, I wanted to expose students to a broader picture by presenting as much as possible quantum field theory as an agnostic tool that can also be used in other contexts, such as cosmology or condensed matter physics. To that effect, I decided to treat topics like the path integral in statistical mechanics, the Schwinger–Keldysh formalism and quantum field theory at finite temperature. Although each of these subjects is well-covered in specialized textbooks, they are usually missing in the more general ones, which may lead to an artificial impression of segmentation of the field.

When the idea to convert my lecture notes into a textbook started to take form, it became obvious that more introductory material was also needed. This led to the addition of the first four chapters, which provide a refresher on basic concepts and set the notations used in the rest of the book. A word of caution is in order regarding the intended role of these introductory chapters: They are meant to make the book sufficiently self-contained for readers who have had prior exposure to quantum field theory, but first-timers will probably find them too fast-paced.

Of course, many topics had to be left out due to a combination of limited space and a lack of familiarity by the author. One of the most prominent absences is supersymmetry, both in its use in extensions of the Standard Model and in toy theories used as playgrounds for formal QFT developments (e.g., supersymmetric Yang–Mills theory). Another topic not treated in this book is the AdS/CFT correspondence, despite it having been a very active subfield over the past 20 years and providing numerous insights about strongly coupled gauge theories.

Each chapter is supplemented by exercises (about 180 in total). Although these exercises range from straightforward (but not necessarily easy) verifications of some technical aspects of the text to more elaborate and time-consuming ones, a general guidance when choosing which exercises to include was the belief that fluency comes by first solving many “simple” exercises in order to master the manipulation of the objects and concepts discussed in the text. Some “recommended” (either because they provide a derivation of an important point of the text, or because they discuss interesting extensions) exercises are marked by a star.

Let me end with a note on the mathematics behind quantum field theory. The subject often carries the reputation of requiring a rather high level of mathematical sophistication. Consequently, some mathematical prerequisites are unavoidable for reading this text comfortably – most notably complex analysis, Fourier transforms, plus some basics of group theory and distribution theory. For completeness, when mathematical tools outside of these areas are necessary, I have included some discussion and sometimes a basic derivation of the tools and results needed to treat the subject at hand.

Acknowledgments

This book would certainly not exist without Stéphane Munier and Pascal Paganini, who offered me the opportunity to teach quantum field theory in the Master's program they are in charge of. Early versions of this manuscript circulated in the form of lecture notes handed to the students, and benefited enormously from their questions, remarks and criticisms. Conversely, this book owes a lot to many talented teachers to whom I have had the pleasure of listening. In particular, Jean-Marie Mercier, Jean-Claude Le Guillou, François Delduc and Patrick Aurenche had a long-lasting impact on how I think about physics in general and quantum field theory in particular.

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