CHAPTER 1

Basics of Quantum Field Theory

1.1 Why Quantum Field Theory?

After the revolutions of relativity and quantum mechanics at the beginning of the twentieth century, theoretical physics seemed to rest on two pillars that had few connections with each other. On the one hand, one had a very successful non-quantum theory of electromagnetic radiation that had relativistic invariance built in, and on the other hand quantum mechanics provided a very effective way of predicting the energy spectrum of a particle in a given external potential but was obviously not relativistically covariant. This situation was not fully satisfactory but one could live with it, provided certain questions are not asked.

Simple estimates tell us that the electron in the hydrogen atom is not relativistic. Indeed, the energy of the electron in the ground state is $E_0 = m_e e^4/(32\pi^2\hbar^2\epsilon_0^2) \approx 13.6$ eV, while its mass is $m_e \approx 0.5$ meV$/c^2$. From this energy, we may estimate the ratio of the electron velocity to the speed of light by $(v_e/c)^2 \sim E_0/(m_e c^2) = \alpha^2/2$, where $\alpha = e^2/(4\pi\epsilon_0\hbar c) \approx 1/137$ is a dimensionless constant – called the fine structure constant – that encodes the strength of the electromagnetic interactions that bind the electron to the hydrogen nucleus (α is proportional to the product of the electrical charges of the electron and of the hydrogen nucleus). If α was much larger,[1] the energy of the electron would get closer to $m_e c^2$, and we would expect relativistic corrections to become non-negligible. Moreover, one may view the non-relativistic framework of quantum mechanics as the zeroth-order approximation of a more general expansion in powers of v_e/c. Since the dimensionless constant α contains a factor c^{-1}, the expansion in v_e/c is also an expansion in α. Even if these corrections are very small, they should exist in principle (in fact, the fine structure of atomic spectra, known since 1887,

[1] Experimentally, this could occur with atoms of high Z that are highly ionized, i.e., stripped of most of their electrons. In this case, α would be replaced by $Z\alpha$.

has later been interpreted as such a relativistic correction), but their calculation certainly requires incorporating some ingredients of special relativity into the framework of quantum mechanics.

The quantum mechanical setup for calculating the spectrum of the hydrogen atom uses the Coulomb potential of the hydrogen nucleus. In this framework, this potential simply acts as a background in which the dynamics of the electron take place, but it does not play an active role in the problem. It would be much more satisfactory to have a theoretical framework in which matter and radiation are treated on the same footing, especially given the fact that quantum mechanics completely blurs the frontier between waves and particles. This unsatisfactory dichotomy arises in the excitation (respectively, de-excitation) of an atom by absorption (respectively, emission) of radiation. But since photons are massless, any framework that incorporates them on the same footing as the electron must be relativistic. Another place where this classical treatment of electromagnetic fields is lacking is in the very concept of a point-like electron. If one imagines reaching it as the zero radius limit of a spherical charge distribution, one can define the electrostatic energy contained in this sphere (defined as the energy necessary to bring these charges from infinity into the sphere). This energy becomes infinite when the radius goes to zero (and becomes comparable to the rest energy $m_e c^2$ of the electron at a finite radius known as the classical radius of the electron).

Quantum mechanics is all about measurements, whose expectations are calculated as averages of operators in the state vector of the system. In this context, one could imagine a "composite" measurement that consists of two local measurements performed at space-time points with a space-like separation. Special relativity tells us that, because no signal may propagate faster than the speed of light, these two measurements are not causally connected and their results should be independent. In quantum mechanics, the independence of two measurements is encoded in the fact that the corresponding operators commute. However, the notion that space-like separated local operators should commute is not naturally present in quantum mechanics. Therefore, we anticipate that it makes predictions that are not fully consistent with relativity in this type of situation.

Another difficulty with the usual formulation of quantum mechanics is that for each particle in the system under consideration, the wavefunction depends on the position of this particle. This becomes rapidly untractable in systems with more than a few particles. While this issue is just a technical difficulty for non-relativistic systems, it becomes an unsurmountable stumbling block in relativistic systems, for which even the number of particles can change. An obvious example is that of atomic transitions that are accompanied by the emission or absorption of a photon. It should be quite clear that a wavefunction that describes a predefined number of particles cannot accommodate this type of transition. This remark suggests that the framework that brings relativistic covariance into quantum mechanics has to be a somewhat radical departure from the usual formulation of quantum mechanics, at least on a technical level.

Quantum field theory is a theoretical framework that promotes quantum mechanics into a relativistic theory. Historically, the first developed quantum field theory was quantum electrodynamics. However, we shall not start with this example in order to avoid the extra complications related to the non-zero spin of photons and electrons, and to the redundancy due to the gauge invariance of classical electrodynamics. Instead, in the first two chapters, we will introduce the basic concepts of quantum field theory with the example of a *scalar* (i.e., zero spin) field. Although this example has less obvious applications in nature, it has considerable didactical virtues because it allows explanation of the structural aspects of quantum field theory without encumbering the exposition with the extra difficulties posed by spin and gauge invariance. These will be deferred until Chapter 3.

1.2 Special Relativity

1.2.1 Lorentz Transformations

Given these premises, special relativity obviously plays a crucial role in quantum field theory. A major requirement is that various observers in frames that are moving at a constant speed relative to each other should be able to describe physical phenomena using the same laws of physics. This does not imply that the equations governing these phenomena are independent of the observer's frame, but that these equations transform in a constrained fashion – depending on the nature of the objects they contain – under a change of reference frame. This property is called *relativistic covariance*, or *Lorentz covariance*. Let us consider two reference frames $\mathcal{F}$ and $\mathcal{F}'$, in which the coordinates of a given event are respectively x^μ and x'^μ. A *Lorentz transformation* is a linear transformation such that the interval $ds^2 \equiv dt^2 - d\boldsymbol{x}^2$ is the same in the two frames.[2] If we denote the coordinate transformation by

$$x'^\mu = \Lambda^\mu{}_\nu \, x^\nu, \tag{1.1}$$

the matrix Λ of the transformation must obey

$$g_{\mu\nu} \, \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}, \tag{1.2}$$

where $g_{\mu\nu}$ is the Minkowski metric tensor $g_{\mu\nu} \equiv \mathrm{diag}\,(1,-1,-1,-1)$. When equipped with the composition law, the set of Lorentz transformations becomes a group, the *Lorentz group*. From eq. (1.2), we can see that the inverse of a Lorentz transformation is given by

$$\Lambda_\mu{}^\nu = \left(\Lambda^{-1}\right)_\nu{}^\mu. \tag{1.3}$$

Infinitesimal Lorentz transformations are those that relate reference frames that have a very small relative velocity. They can be written as a small deviation about the identical transformation,

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \tag{1.4}$$

with all components of ω much smaller than unity. The definition of Lorentz transformations implies that $\omega_{\mu\nu} = -\omega_{\nu\mu}$ (with all indices down). Consequently, there are six independent Lorentz transformations, three of which are ordinary *rotations* and three are *boosts*. Note that the infinitesimal transformations (1.4) have a determinant[3] equal to $+1$ (they are called *proper* transformations), and do not change the direction of the time axis since $\Lambda^0{}_0 \approx 1 \geq 0$ (they are called *orthochronous*). Any combination of such infinitesimal transformations shares the same properties, and their set forms a subgroup of the Lorentz group.

[2]The physical premises of special relativity require that the speed of light be the same in all inertial frames, which implies solely that $ds^2 = 0$ be preserved in all inertial frames. The group of transformations that achieves this is called the *conformal group*. In four space-time dimensions, the conformal group is 15 dimensional, and in addition to the six orthochronous Lorentz transformations it contains translations, dilatations, as well as nonlinear transformations called *special conformal transformations* (see Exercise 12.9).

[3]From eq. (1.2), the determinant may be equal to ± 1.

1.2.2 Lorentz and Poincaré Algebras

In the previous section, we introduced Lorentz transformations via their action on the coordinates x^μ. But of course, coordinates are not the only objects that vary when changing the reference frame. For instance, any tensor transforms as

$$T'^{\mu\nu\cdots} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \cdots T^{\alpha\beta\cdots}. \tag{1.5}$$

In addition, in a quantum system, a Lorentz transformation Λ should also act on the states in the Hilbert space via a linear transformation $U(\Lambda)$,

$$\big|\boldsymbol{\alpha}_{\mathcal{F}'}\big\rangle = U(\Lambda)\,\big|\boldsymbol{\alpha}_{\mathcal{F}}\big\rangle, \tag{1.6}$$

that forms a representation of the Lorentz group, i.e.,

$$U(\Lambda'\Lambda) = U(\Lambda')U(\Lambda). \tag{1.7}$$

This property simply means that under a succession of two Lorentz transformations Λ and Λ', the resulting state can either be obtained in a single transformation corresponding to the product of the Lorentz transformations, or in a two-step process in which the two transformations are applied successively. For an infinitesimal Lorentz transformation, we can write

$$U(1+\omega) = 1 + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + \mathcal{O}(\omega^2). \tag{1.8}$$

(The prefactor $i/2$ in the second term of the right-hand side is conventional.) Since the $\omega_{\mu\nu}$ are antisymmetric, the $M^{\mu\nu}$ can also be chosen as antisymmetric. The $M^{\mu\nu}$ are called the *generators* of the Lorentz group in the representation U. By using eq. (1.7) for the Lorentz transformation $\Lambda^{-1}\Lambda'\Lambda$, we arrive at

$$U^{-1}(\Lambda)M^{\mu\nu}U(\Lambda) = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma M^{\rho\sigma}, \tag{1.9}$$

indicating that $M^{\mu\nu}$ transforms as a rank-2 tensor. When used with an infinitesimal transformation $\Lambda = 1+\omega$, this identity leads (see Exercise 1.1) to the commutation relation that defines the Lie algebra of the Lorentz group,

$$\big[M^{\mu\nu}, M^{\rho\sigma}\big] = i(g^{\mu\rho}M^{\nu\sigma} - g^{\nu\rho}M^{\mu\sigma}) - i(g^{\mu\sigma}M^{\nu\rho} - g^{\nu\sigma}M^{\mu\rho}). \tag{1.10}$$

When necessary, it is possible to divide the six generators $M^{\mu\nu}$ into three generators J^i for ordinary spatial rotations, and three generators K^i for the Lorentz boosts along each of the spatial directions:

$$\begin{aligned} &\text{Rotations:} && J^i \equiv \tfrac{1}{2}\epsilon_{ijk}\,M^{jk}, \\ &\text{Lorentz boosts:} && K^i \equiv M^{i0}, \end{aligned} \tag{1.11}$$

where ϵ_{ijk} is the three-dimensional Levi-Civita symbol[4] normalized by $\epsilon_{123} = +1$ (thus, $J^1 = M^{23}, J^2 = M^{31}, J^3 = M^{12}$).

[4]Throughout this book, the Levi-Civita symbol on a set of ordered indices $\{i_1, i_2, \cdots, i_n\}$ is consistently defined by $\epsilon_{i_1 i_2 \cdots i_n} = +1$, *with lowered indices*. In circumstances where it makes sense to raise the indices, this is done as usual by multiplication with the metric tensor.

Poincaré group: The group of Lorentz transformations can be extended by adding the translations, resulting in a larger group of transformations known as the *Poincaré group*. A translation is parameterized by the 4-vector a^μ by which all coordinates are shifted, $x^\mu \to x^\mu + a^\mu$. A generic transformation for the Poincaré group is thus a pair (a, Λ) such that

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu. \tag{1.12}$$

(In this definition, the Lorentz transformation is applied first.) As for the Lorentz transformations, the action of infinitesimal translations on states can be represented by

$$U(a) = 1 + i\, a_\mu P^\mu + \mathcal{O}(a^2), \tag{1.13}$$

where P^μ is the generator of translations (which is nothing but the 4-momentum operator). In a fashion similar to eq. (1.9), we obtain

$$U^{-1}(\Lambda) P^\mu U(\Lambda) = \Lambda^\mu{}_\rho P^\rho, \tag{1.14}$$

which leads to the following commutation relation (see Exercise 1.2) between P^μ and $M^{\mu\nu}$,

$$\begin{aligned} \big[P^\mu, M^{\rho\sigma}\big] &= i(g^{\mu\sigma}P^\rho - g^{\mu\rho}P^\sigma), \\ \big[P^\mu, P^\nu\big] &= 0. \end{aligned} \tag{1.15}$$

Let us illustrate on a simple example how to relate the measurements of the momentum in a certain state performed by two observers in different reference frames. The two observers measure the expectation values $\big\langle \boldsymbol{\alpha}_{\mathcal{F}}\big|P^\mu\big|\boldsymbol{\alpha}_{\mathcal{F}}\big\rangle$ and $\big\langle \boldsymbol{\alpha}_{\mathcal{F}'}\big|P^\mu\big|\boldsymbol{\alpha}_{\mathcal{F}'}\big\rangle$. These expectation values are related by

$$\big\langle \boldsymbol{\alpha}_{\mathcal{F}'}\big|P^\mu\big|\boldsymbol{\alpha}_{\mathcal{F}'}\big\rangle = \big\langle \boldsymbol{\alpha}_{\mathcal{F}}\big|U^{-1}(\Lambda)\, P^\mu\, U(\Lambda)\big|\boldsymbol{\alpha}_{\mathcal{F}}\big\rangle = \Lambda^\mu{}_\nu \big\langle \boldsymbol{\alpha}_{\mathcal{F}}\big|P^\nu\big|\boldsymbol{\alpha}_{\mathcal{F}}\big\rangle. \tag{1.16}$$

Unsurprisingly, since the two observers measure the 4-momentum of the same system in two different frames, the results of their measurements are related in a simple way by the Lorentz transformation of a vector.

1.2.3 One-Particle States

Let us denote $\big|\mathbf{p}, \sigma\big\rangle$ as a one-particle state, where $\mathbf{p}$ is the 3-momentum of that particle, and σ denotes its other quantum numbers. Since this state contains a particle with a definite momentum, it is an eigenstate of the momentum operator P^μ, namely

$$P^\mu \big|\mathbf{p}, \sigma\big\rangle = p^\mu \big|\mathbf{p}, \sigma\big\rangle, \quad \text{with } p^0 \equiv \sqrt{\mathbf{p}^2 + m^2}. \tag{1.17}$$

Let us now act on this state with a Lorentz transformation, to obtain $U(\Lambda)\big|\mathbf{p}, \sigma\big\rangle$. We have

$$P^\mu\, U(\Lambda)\big|\mathbf{p}, \sigma\big\rangle = U(\Lambda) \underbrace{U^{-1}(\Lambda) P^\mu U(\Lambda)}_{\Lambda^\mu{}_\nu P^\nu} \big|\mathbf{p}, \sigma\big\rangle = \Lambda^\mu{}_\nu p^\nu\, U(\Lambda)\big|\mathbf{p}, \sigma\big\rangle. \tag{1.18}$$

Therefore, $U(\Lambda)\big|\mathbf{p}, \sigma\big\rangle$ is an eigenstate of momentum with eigenvalue $(\Lambda p)^\mu$, and we may write it as a linear combination of all the states with momentum Λp,

$$U(\Lambda)\big|\mathbf{p}, \sigma\big\rangle = \sum_{\sigma'} C_{\sigma\sigma'}(\Lambda; \mathbf{p})\, \big|\Lambda\mathbf{p}, \sigma'\big\rangle. \tag{1.19}$$

1.2.4 Little Group

Consider a momentum p^μ such that $p^0 > 0$ and $p^2 = m^2$ (it is said to be *on-shell*). Any such vector can be obtained by applying an orthochronous Lorentz transformation to some reference momentum q^μ located on the same mass-shell (i.e., $q^0 > 0, q^2 = m^2$),

$$p^\mu \equiv L^\mu{}_\nu(\mathbf{p})\, q^\nu. \tag{1.20}$$

The choice of the reference 4-vector is not important, but depends on whether the particle under consideration is massive or not. Convenient choices are the following:

- $m > 0$: $q^\mu \equiv (m, 0, 0, 0)$, the 4-momentum of a massive particle at rest;
- $m = 0$: $q^\mu \equiv (\omega, 0, 0, \omega)$, the 4-momentum of a massless particle moving in the third direction of space.

Then, we may define generic one-particle states from those corresponding to the reference momentum as follows:

$$\big|\mathbf{p}, \sigma\big\rangle \equiv \mathcal{N}_\mathbf{p}\, U(L(\mathbf{p}))\big|\mathbf{q}, \sigma\big\rangle, \tag{1.21}$$

where $L(\mathbf{p})$ is the Lorentz transformation that transforms q^μ into p^μ and $\mathcal{N}_\mathbf{p}$ is a numerical prefactor that may be necessary to properly normalize the states. This definition leads to

$$U(\Lambda)\big|\mathbf{p}, \sigma\big\rangle = \mathcal{N}_\mathbf{p}\, U\big(L(\Lambda\mathbf{p})\big)\, U\big(\underbrace{L^{-1}(\Lambda\mathbf{p})\Lambda L(\mathbf{p})}_{\Sigma}\big)\big|\mathbf{q}, \sigma\big\rangle. \tag{1.22}$$

Note that the Lorentz transformation $\Sigma \equiv L^{-1}(\Lambda\mathbf{p})\Lambda L(\mathbf{p})$ maps q^μ to itself,

$$q^\mu \underset{L(\mathbf{p})}{\longrightarrow} p^\mu \underset{\Lambda}{\longrightarrow} (\Lambda p)^\mu \underset{L^{-1}(\Lambda\mathbf{p})}{\longrightarrow} q^\mu, \tag{1.23}$$

and therefore belongs to the subgroup of the Lorentz group that leaves q^μ invariant, called the *little group* of q^μ. Thus, when $U(\Sigma)$ acts on the reference state, the momentum remains unchanged and only the other quantum numbers may vary:

$$U(\Sigma)\,\big|\mathbf{q}, \sigma\big\rangle = \sum_{\sigma'} C_{\sigma\sigma'}(\Sigma)\,\big|\mathbf{q}, \sigma'\big\rangle. \tag{1.24}$$

Moreover, the coefficients $C_{\sigma\sigma'}(\Sigma)$ in the right-hand side of this formula define a representation of the little group,

$$C_{\sigma\sigma'}(\Sigma_2\Sigma_1) = \sum_{\sigma''} C_{\sigma\sigma''}(\Sigma_2)\, C_{\sigma''\sigma'}(\Sigma_1). \tag{1.25}$$

Massive particles: In the case of a massive particles, the little group is made of the Lorentz transformations that leave the vector $q^\mu = (m, 0, 0, 0)$ invariant, which is the group of all rotations in three-dimensional space, $SO(3)$. The additional quantum number σ is therefore a label that enumerates the possible states in a given representation of the rotation group. These representations correspond to the angular momentum, but since we are in the rest frame of the particle, this is also its spin. For a spin s, the dimension of the representation is $2s + 1$, and σ takes the values $-s, 1 - s, \cdots, +s$.

Massless particles: In the massless case, we look for Lorentz transformations $\Sigma^\mu{}_\nu$ that leave $q^\nu = (\omega, 0, 0, \omega)$ invariant. For an infinitesimal transformation, $\Sigma^\mu{}_\nu \approx \delta^\mu{}_\nu + \omega^\mu{}_\nu$, this gives the following general form:

$$\omega_{\mu\nu} = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & 0 \\ -\alpha_1 & 0 & -\theta & \alpha_1 \\ -\alpha_2 & \theta & 0 & \alpha_2 \\ 0 & -\alpha_1 & -\alpha_2 & 0 \end{pmatrix}, \tag{1.26}$$

where $\alpha_{1,2}, \theta$ are three real infinitesimal parameters. Therefore, an infinitesimal transformation $U(\Sigma)$ reads

$$U(\Sigma) \approx 1 - i\theta \underbrace{M^{12}}_{J^3} - i\alpha_1 \underbrace{(M^{10} + M^{31})}_{K^1 + J^2 \equiv B^1} - i\alpha_2 \underbrace{(M^{20} - M^{23})}_{K^2 - J^1 \equiv B^2}. \tag{1.27}$$

Thus, the little group for massless particles is three-dimensional, with generators J^3 (the projection of the angular momentum in the direction of the momentum) and the combinations $B^{1,2}$.[5] Using eq. (1.10), we have

$$\big[J^3, B^1\big] = i\,B^2, \quad \big[J^3, B^2\big] = -i\,B^1, \quad \big[B^1, B^2\big] = 0. \tag{1.28}$$

The last commutator implies that we may choose states that are simultaneous eigenstates of B^1 and B^2. However, non-zero eigenvalues for $B^{1,2}$ may be shown to lead to a continuum of states with the same momentum, which is not realized in nature. Therefore, the only eigenvalue that labels the massless states is that of J^3, that generates rotations about the direction of momentum,

$$J^3\,\big|\mathbf{q},\sigma\big\rangle = \sigma\,\big|\mathbf{q},\sigma\big\rangle, \quad U(\Sigma)\,\big|\mathbf{q},\sigma\big\rangle \underset{\alpha_{1,2}=0}{=} e^{-i\sigma\theta}\,\big|\mathbf{q},\sigma\big\rangle. \tag{1.29}$$

The number σ is called the *helicity* of the particle. After a rotation of angle $\theta = 2\pi$, the state must return to itself (bosons) or its opposite (fermions), implying that the helicity must be a half-integer,

$$\text{bosons:}\quad \sigma = 0, \pm 1, \pm 2, \cdots, \qquad \text{fermions:}\quad \sigma = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \cdots \tag{1.30}$$

1.3 Free Scalar Fields, Mode Decomposition

1.3.1 Scalars and Scalar Fields

In special relativity, a *scalar* quantity is any quantity invariant under a Lorentz transformation. One may think of a scalar as simply being a plain number. A *scalar field* extends this notion

[5]The generators $B^{1,2}$ are the generators of *Galilean boosts* in the (x^1, x^2) plane transverse to the particle momentum, i.e., the transformations that shift the transverse velocity, $v^j \to v^j + \delta v^j$. The physical reason for their appearance in the discussion of massless particles is time dilation: in the observer's frame, the transverse dynamics of a particle moving at the speed of light is infinitely slowed down by time dilation, and is therefore non-relativistic (this intuitive idea can be further substantiated by light-cone quantization – see Exercise 1.10).

to functions of space–time: $\phi(x)$ is a scalar field operator if it is invariant under a Lorentz transformation, except for the change of coordinate induced by the transformation:

$$U^{-1}(\Lambda)\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x). \tag{1.31}$$

This formula just reflects the fact that the point x where the transformed field is evaluated was located at the point $\Lambda^{-1}x$ before the transformation. The simplicity of this transformation law is the reason why we start our study of quantum field theory with scalar fields. As we shall see, scalar fields describe particles that have no other quantum number besides their momentum, i.e., spin-0 particles (also called scalar particles). The first derivative $\partial^\mu\phi$ of the field transforms as a 4-vector,

$$U^{-1}(\Lambda)\partial^\mu\phi(x)U(\Lambda) = \Lambda^\mu{}_\nu\partial^{\overline{\nu}}\phi(\Lambda^{-1}x), \tag{1.32}$$

where the bar in $\partial^{\overline{\nu}}$ indicates that we are differentiating with respect to the whole argument of ϕ, i.e., $\Lambda^{-1}x$. Likewise, the second derivative $\partial^\mu\partial^\nu\phi$ transforms like a rank-2 tensor, but the d'Alembertian $\square\phi$ transforms as a scalar.

1.3.2 Quantum Harmonic Oscillators

In order to introduce scalar fields, let us make a detour by a familiar problem in quantum mechanics, that of the harmonic oscillator. But instead of a single oscillator, let us consider a continuous collection of *independent* harmonic oscillators, each of them corresponding to particles with a given momentum $\mathbf{p}$. Each of these harmonic oscillators can be described by a pair of creation and annihilation operators $a^\dagger_{\mathbf{p}}, a_{\mathbf{p}}$, where $\mathbf{p}$ is a 3-momentum that labels the corresponding mode. Note that the energy of the particles is fixed from their 3-momentum by the relativistic dispersion relation,

$$p^0 = E_{\mathbf{p}} \equiv \sqrt{\mathbf{p}^2 + m^2}. \tag{1.33}$$

Harmonic oscillators and free particles: A simple but essential remark is that *independent harmonic oscillators describe a collection of non-interacting particles*. In order to see this, recall that the operators creating or destroying particles with a given momentum $\mathbf{p}$ obey the usual commutation relations,

$$\big[a_{\mathbf{p}}, a_{\mathbf{p}}\big] = \big[a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{p}}\big] = 0, \quad \big[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}\big] \neq 0. \tag{1.34}$$

In contrast, by our assumption that oscillators with different momenta are independent, operators acting on different momenta always commute,

$$\big[a_{\mathbf{p}}, a_{\mathbf{q}}\big] = \big[a^\dagger_{\mathbf{p}}, a^\dagger_{\mathbf{q}}\big] = \big[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}\big] = 0. \tag{1.35}$$

As we shall see shortly, the normalization of the only non-zero commutator can be chosen as follows:

$$\big[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}\big] = (2\pi)^3\, 2E_{\mathbf{p}}\, \delta(\mathbf{p} - \mathbf{q}). \tag{1.36}$$

The independence between the momenta also implies that the Hamiltonian of this system is a sum of the Hamiltonians of independent harmonic oscillators, which we choose to normalize as follows:[6]

$$\mathcal{H} = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}}\, E_\mathbf{p}\, \left(a^\dagger_\mathbf{p} a_\mathbf{p} + V\,E_\mathbf{p}\right), \tag{1.37}$$

where V is the volume of the system. The term $V\,E_\mathbf{p}$ simply shifts all the energy levels of the system by a constant, but energy differences are unaffected by this term. We have included it nevertheless in order to facilitate the contact with the usual form[7] of the Hamiltonian of harmonic oscillators in quantum mechanics. Note that once we have chosen the normalization of the creation and annihilation operators via eq. (1.36), the Hamiltonian is completely constrained. Indeed, we can now check that

$$\left[\mathcal{H}, a^\dagger_\mathbf{p}\right] = +E_\mathbf{p}\, a^\dagger_\mathbf{p}, \quad \left[\mathcal{H}, a_\mathbf{p}\right] = -E_\mathbf{p}\, a_\mathbf{p}. \tag{1.38}$$

The meaning of this equation is as follows: When $a^\dagger_\mathbf{p}$ acts on an energy eigenstate of energy E, the result is another energy eigenstate, of energy $E + E_\mathbf{p}$. This property is equivalent to the statement that particles are non-interacting, since it tells us that adding a particle of momentum $\mathbf{p}$ does not affect the rest of the system. In other words, a system with N particles has no binding energy.

Occupation number: In order to gain more intuition about the Hamiltonian (1.37), it is useful to introduce the *occupation number* $f_\mathbf{p}$ defined by

$$2E_\mathbf{p}\, V\, f_\mathbf{p} \equiv a^\dagger_\mathbf{p} a_\mathbf{p}. \tag{1.39}$$

In terms of $f_\mathbf{p}$, the above Hamiltonian reads

$$\mathcal{H} = V \int \frac{d^3\mathbf{p}}{(2\pi)^3}\, E_\mathbf{p}\, \left(f_\mathbf{p} + \tfrac{1}{2}\right). \tag{1.40}$$

The expectation value of $f_\mathbf{p}$ has the interpretation of the number of particles per unit of phase-space (i.e., per unit of volume in coordinate space and per unit of volume in momentum space), and the 1/2 in $f_\mathbf{p} + \frac{1}{2}$ is the ground state occupation of each oscillator. This is

[6] The measure $d^3\mathbf{p}/(2\pi)^3 2E_\mathbf{p}$ is Lorentz invariant. Moreover, it emerges naturally from the four-dimensional momentum integration $d^4p/(2\pi)^4$ constrained by the positive energy mass-shell condition $2\pi\,\theta(p^0)\,\delta(p^2 - m^2)$.

[7] In relativistic quantum field theory, it is customary to use a system of units, called *natural units*, in which $\hbar = 1$, $c = 1$, $\epsilon_0 = 1$ (and also $k_B = 1$ when the Boltzmann constant is needed to relate energies and temperature). In this system of units, the action $\mathcal{S}$ is dimensionless. Mass, energy, momentum and temperature have the same dimension, which is the inverse of the dimension of length and duration:

$$[\text{mass}] = [\text{energy}] = [\text{momentum}] = [\text{temperature}] = [\text{length}^{-1}] = [\text{duration}^{-1}]\ .$$

Moreover, in four dimensions, the creation and annihilation operators introduced in eq. (1.37) have the dimension of an inverse energy:

$$[a_\mathbf{p}] = [a^\dagger_\mathbf{p}] = [\text{energy}^{-1}]$$

(the occupation number $f_\mathbf{p}$ defined in eq. (1.39) is dimensionless).

reminiscent of the fact that the energy of the level n of a quantized harmonic oscillator of base energy ω is $E_n = (n+\frac{1}{2})\omega$.

Ground state: The ground state of the Hamiltonian (1.37) is the empty state, in which the expectation values of number operators are zero, $\big\langle 0\big|a^\dagger_{\mathbf{p}} a_{\mathbf{p}}\big|0\big\rangle = 0$. This state, also called the *vacuum*, is usually denoted $\big|0\big\rangle$. The vacuum of a theory is invariant under Lorentz transformation, $U(\Lambda)\,\big|0\big\rangle = \big|0\big\rangle$. The physical meaning of this property is that the vacuum state appears empty to all observers that are moving at constant velocities[8] relative to each other. Let us now consider one-particle states, obtained by acting on the vacuum with a single creation operator,

$$\big|\mathbf{p}\big\rangle \equiv a^\dagger_{\mathbf{p}}\,\big|0\big\rangle. \tag{1.41}$$

The standard notation for states populated with a few particles is simply to list the momenta (and their other quantum numbers, if applicable) contained in the state. Since scalar states have no other quantum numbers, we have

$$a^\dagger_{\Lambda\mathbf{p}}\,\big|0\big\rangle = \big|\Lambda\mathbf{p}\big\rangle = U(\Lambda)\,\big|\mathbf{p}\big\rangle = \underbrace{U(\Lambda)\,a^\dagger_{\mathbf{p}}\,U^{-1}(\Lambda)}_{a^\dagger_{\Lambda\mathbf{p}}}\,\underbrace{U(\Lambda)\,\big|0\big\rangle}_{\big|0\big\rangle}, \tag{1.42}$$

from which we read off the action of $U(\Lambda)$ on the creation operators. For instance, if two observers in frames $\mathcal{F}$ and $\mathcal{F}'$ measure the occupation number in a state $\boldsymbol{\alpha}$, their respective measurements will be

$$\begin{aligned} f_{\mathbf{p}}(\boldsymbol{\alpha}_{\mathcal{F}}) &= \frac{\big\langle \boldsymbol{\alpha}_{\mathcal{F}}\big|a^\dagger_{\mathbf{p}} a_{\mathbf{p}}\big|\boldsymbol{\alpha}_{\mathcal{F}}\big\rangle}{2E_{\mathbf{p}}V},\\ f_{\mathbf{p}}(\boldsymbol{\alpha}_{\mathcal{F}'}) &= \frac{\big\langle \boldsymbol{\alpha}_{\mathcal{F}'}\big|a^\dagger_{\mathbf{p}} a_{\mathbf{p}}\big|\boldsymbol{\alpha}_{\mathcal{F}'}\big\rangle}{2E_{\mathbf{p}}V} = \frac{\big\langle \boldsymbol{\alpha}_{\mathcal{F}}\big|U^{-1}(\Lambda)\,a^\dagger_{\mathbf{p}} a_{\mathbf{p}}\,U(\Lambda)\big|\boldsymbol{\alpha}_{\mathcal{F}}\big\rangle}{2E_{\mathbf{p}}V} = f_{\Lambda^{-1}\mathbf{p}}(\boldsymbol{\alpha}_{\mathcal{F}}). \end{aligned} \tag{1.43}$$

(The factor $E_{\mathbf{p}}V$ is Lorentz invariant.) As expected, the observer in $\mathcal{F}'$ measures at $\mathbf{p}$ the same occupation number as the observer in $\mathcal{F}$ at the momentum $\Lambda^{-1}\mathbf{p}$.

1.3.3 Scalar Field Operator

At this point, we have a collection of non-interacting quantum oscillators, one for each possible momentum $\mathbf{p}$, that describe a system of non-interacting particles. Before we turn to something more useful with interactions, let us first rephrase the description of this non-interacting system in a way that is more explicitly Lorentz covariant.

[8]For this to hold, it is important that there is no relative acceleration. A state that appears to be empty in one frame may appear populated in an accelerating frame, a phenomenon known as the *Unruh effect*.