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Introduction and Problem Formulation

Mathematics is the key and door to the sciences

– Galileo Galilei (1564–1642)

1.1 History, Background and Rationale

In examining the dynamics of any physical system, the concept of stability becomes relevant only after first establishing the possibility of equilibrium. Once this step has been taken, the concept of stability becomes pervasive, regardless of the actual system being probed. As expressed by Betchov & Criminale (1967), stability can be defined as the ability of a dynamical system to be immune to small disturbances. It is clear that the disturbances need not necessarily be small in magnitude and therefore may become amplified. As such, there is a departure from the state of equilibrium. Should no equilibrium be possible, then it can already be concluded that the particular system in question is statically unstable and the dynamics is a moot point.

Such tests for stability can be and are made in any field, such as mechanics, astronomy, electronics and biology, for example. In each case from this list, there is a common thread in that only a finite number of discrete degrees of freedom are required to describe the motion and there is only one independent variable. Like tests can be made for problems in continuous media but the number of degrees of freedom becomes infinite and the governing equations are now partial differential equations instead of the ordinary variety. Thus, conclusions are harder to obtain in any general manner, but it is not impossible. In fact, successful analysis of many such systems has been made and this has been particularly true in fluid mechanics. This premise is even more so today because there are far more advanced means of computation available to supplement analytical techniques. Likewise the means for experimentation has improved in profound ways and will be highlighted throughout the text in validation of the theoretical and computational results.

Fundamentally, there is no difficulty in presenting the problem of stability in fluid mechanics. The governing Navier–Stokes continuum equations for the conservation of momentum and mass that is often expressed by constraints, such as incompressibility that requires the fluid velocity to be solenoidal in a somewhat general sense, are the tools of the science. A specific flow is then fully determined by satisfying the boundary conditions that must be met for that flow. Other considerations involve the importance of the choice of the coordinate system that is best to describe the flow envisioned and whether or not there is any body force, say. Then, the important first step is to identify a flow that is in equilibrium. For this purpose, a flow that is in equilibrium need not necessarily be time independent, but the system is no longer accelerated due to the balance of all forces. For such flows meeting these conditions very few, if any, remain that have not been theoretically evaluated using this approach, but, because the governing equations of motion are a set of nonlinear partial differential equations, the results are most often the result of approximations. Nevertheless these flows are well established, many have been experimentally confirmed, and they are all laminar. In addition, a few exact solutions of the governing equations are known. In such cases, where more complex physics is entailed, such as compressibility or electrical conductivity of the fluid, similar arguments can be made and results have been equally obtainable.

Essentially there are three major categories of base mean flows, namely: (a) flows that are parallel or almost parallel; (b) flows with curved streamlines and; (c) flows where the mean flow has a zero value. Examples of the parallel variety are channel flows, such as plane Couette and Poiseuille flows where the flows are confined by two solid boundaries. There is one mean component for the mean velocity and it is a function of the coordinate that defines the locations of the boundaries. In a polar coordinate system, pipe flow is another example of note. Almost parallel flows are of two main categories: (i) free shear flows, such as the jet, wake and mixing layer where there are no solid boundaries in the flow and (ii) the flat-plate boundary layer where there is but one solid boundary. In these terms, (i) and (ii) have two components for the mean velocity, and they are both functions of the coordinate in the direction of the flow as well as the one that defines the extent of the flow. In Cartesian terms, if U and V are the mean velocity components in the x and y spatial directions, respectively, then almost parallel assumes that $V \ll U$ and that the variation of U with respect to the downstream variable x is weak. Group (b) has flows such as that between concentric circular cylinders (Taylor problem) or flow on concave walls (Görtler problem). The cases where there is no mean flow (e.g., Rayleigh problem, Bénard cells) are simply special cases of the more general picture. Whether from the point of view of the physics or the mathematics needed to make analyses, each of these prototypes has its own unique features and it is the stability of the system that is the question to be answered. It should be clear that the actual causes of any resulting instability will vary as well.

It should be again stressed that, regardless of the methods required for obtaining any mean flow, they are laminar and are in equilibrium or near equilibrium. But, unfortunately, just as the adage states, "turbulence is the rule and not the exception to fluid motion." In other words, laminar flows are extremely hard to maintain; transition to turbulence will occur in the short or the long time. One need only to observe the flow over the wings of an airplane, the meandering of a river, the outflow from the garden hose or the resulting flow behind bluff bodies in both the laboratory and in nature to witness this predominance first hand. Laminar flow is orderly, can be well predicted and is most generally desired. The illustrations of Figs. 1.1, 1.2 and 1.3 vividly demonstrate the more-than-subtle differences for these two flows in the boundary layer setting. Benefits of laminar flow include less drag and reduced acoustics when compared to the turbulent state. Figure 1.1 shows the clean streamline pattern over a flat plate, reminiscent of laminar flow, whereas Fig. 1.2 shows the random turbulent boundary layer over a segment of the same flat plate. Although transition occurs via a different mechanism on a rotating cone, Fig. 1.3 shows the entire set of fluid states whereby the flow is laminar at the apex of the cone. The focus of this text becomes clear as the flow is disturbed and "transitions" to a state between laminar and turbulent. Finally, the flow is fully random, chaotic or what is called turbulent. Contrary to the benefits of laminar flow, a case where a benefit from turbulent flow would be desired over laminar is mixing, for example. The goal of predicting or even approximating the process of transition has been a stated goal throughout the history of fluid mechanics and, it was once thought, stability analysis would be able to do this. Any success has been limited but stability analysis can explain – for almost all of the major cases – why a basic flow cannot be maintained indefinitely.

Although the main focus of the text is on the mathematics of predicting flow instabilities, the classical experiments of Reynolds (1883) are introduced in Fig. 1.4, which shows the circular pipe flow experiment. Note the very raw experimental setup of the era compared with modern-day more advanced laboratory systems. Figure 1.5 shows the classical experiment due for flow in a circular pipe whereby dye was inserted and the mean flow run at different values through a number of pipe diameters. This was an extremely important series of experiments to modern day fluid mechanics, so it is worth revisiting

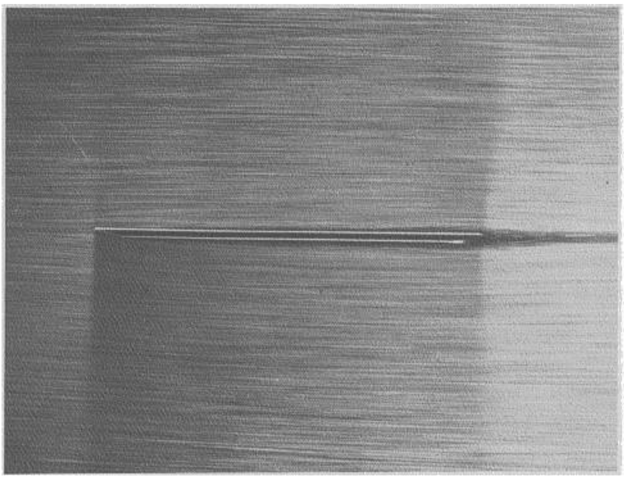

Figure 1.1 Laminar boundary layer on a flat plate (Werlé, 1974).

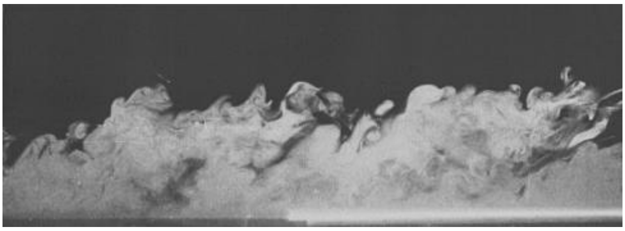

Figure 1.2 Turbulent boundary layer on a flat plate (Reprinted from Falco, 1977 with the permission of AIP Publishing).

these results briefly, as well as the thoughts of Osborne Reynolds. At the beginning of his paper, Reynolds stated the following:

> There appeared to be two ways of proceeding – the one theoretical, the other practical. The theoretical method involved the integration of the equations for unsteady motion in a way that had not been accomplished and which, considering the general intractability of the equations, was not promising. The practical method was to test the relation between U, μ/ρ, and c.

The first way of proceeding – theory – is the primary focus of this text and clearly shows the progress made over time and, with the advent of computers, the equations have become tractable. The second way of proceeding – namely, experimentation – was important to the contemporary scientist because the variation of velocity U, kinematic viscosity μ/ρ, and pipe radius c was the advent of the Reynolds number $Re = \rho U c/\mu$.

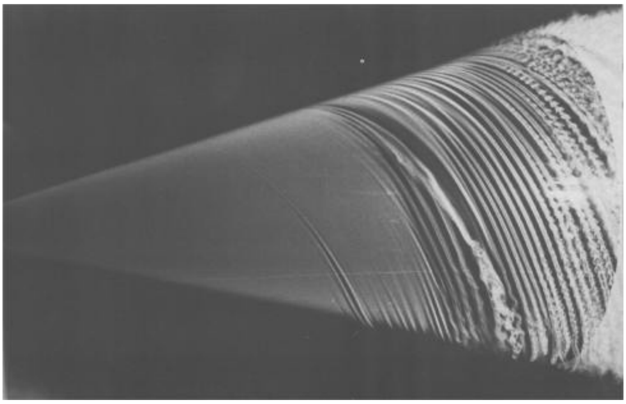

Figure 1.3 Spiral vortices on a cone in rotation with freestream (Kobayashi, Kohama & Kurosawa, 1983, reproduced with permission).

In returning to the discussion of Reynolds' main observations in Fig. 1.4, the original organized parallel laminar flow is seen at several stages with the ultimate breakdown and fully random three-dimensional motion transpiring. At low "Reynolds number," the dye is transported through the pipe evident as a straight line at the top of the image. As the Reynolds number increases, or a critical velocity is reached, Reynolds noted:

> And it was a matter of surprise to me to see the sudden force with which the eddies sprang into existence, showing a highly unstable condition to have existed at the time the steady motion broke down.

As the critical velocity increases, the dye image clearly shows a more random or turbulent pattern. Ironically, this problem is one where stability theory has not been able to make any conclusions whatsoever and remains an enigma in the field. In short, linear theory has been used to investigate this flow in many ways and no solutions that predict instability have been found. This has been found to be true regardless of any added complexities that might be envisioned – for example, axisymmetric versus non-axisymmetric disturbances. Still, it is clear that this flow is unstable.

Drawings of vortices can be traced as far back as those of Leonardo da Vinci that were made in the fifteenth century. The first significant contribution to the theory of hydrodynamic stability is that due to Helmholtz (1868). The principal initial experiments are due to Hagan (1855). Later a major list of contributions can be cited. Reynolds (1883), Kelvin (1880, 1887a,b) and Rayleigh (1879, 1880, 1887, 1892a,b,c, 1895, 1911, 1913, 1914, 1915, 1916a,b) were all ac-

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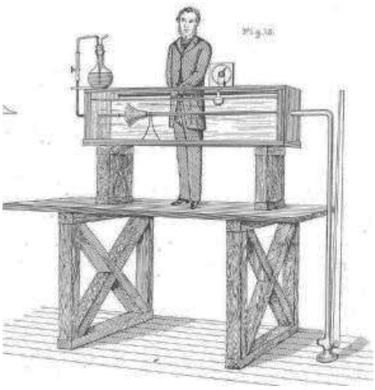

Figure 1.4 Sketch of the Reynolds pipe flow experiment (Reynolds, 1883).

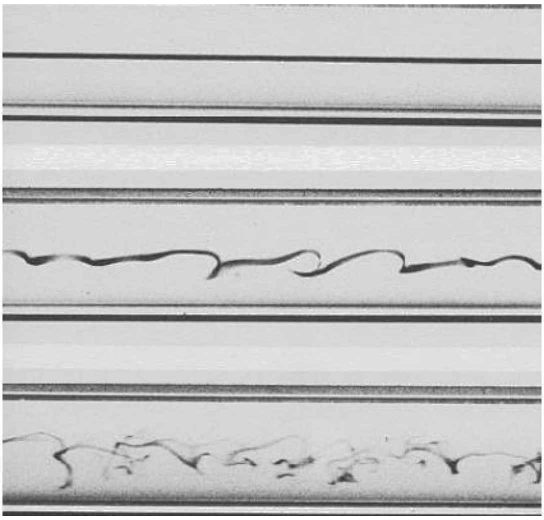

Figure 1.5 Repetition of Reynolds' dye pipe experiment (van Dyke, 1982).

tive in this period. Here, the birth of the Reynolds number as well as the first theorems due to Rayleigh appeared. As has been noted before, Lord Rayleigh was thirty-six when he considered the stability of flames and then published his work on jets. At seventy-two he began to do work in nonlinear stability theory! Unlike Reynolds' pipe experiment, which was intrinsically viscous, the exceptional theoretical work of Kelvin and Rayleigh was done using the inviscid approximation in the analysis.

Independently, Orr (1907a,b) and Sommerfeld (1908) framed the viscous stability problem. Both were attempting to investigate channel flow, with Orr considering plane Couette flow, and Sommerfeld plane Poiseuille flow. Of course one case is the limit of the other and the combination has led to the Orr–Sommerfeld equation that has become the essential basis in the theory of hydrodynamic stability. But, even here, it should be remembered that it was not until twenty-two years after the derivation of this equation that any solution at all could be produced. Tollmien (1929) calculated the first neutral eigenvalues for plane Poiseuille flow and showed that there was a critical value for the Reynolds number. This work was made possible by the development of Tietjens' functions (Tietjens, 1925) and analysis of Heisenberg (1924), connected with the topic of resistive instability. Romanov (1973) proved theoretically that plane Couette flow is stable. Unlike pipe flow, there is no experimental controversy here. Plane Poiseuille flow, on the other hand, is unstable.

Schlichting (1932a,b, 1933a,b,c, 1934, 1935) continued the work of Tollmien and extended it even further. The combination of these efforts have led to the designation for the oscillations that are now the salient results for the stability of parallel or nearly parallel flows, namely Tollmien–Schlichting waves. It should be noted that such waves correspond to those waves where friction is critical and do not exist for any problem that does not include viscosity and are known to be present only in flows where a solid boundary is present in the flow. Also, in the limit of infinite Reynolds number, the flow is stabilized.

Prandtl (1921–1926, 1930, 1935) was active in problems related to stability in the hopes that the theory might lead to the prediction of transition and the onset of turbulence. As mentioned, to date no such success has been achieved but the effort continues as the understanding makes progress. But, for the first time during this period, a major boost to stability analysis was given by the work of Taylor (1923) where theory was confirmed by his experiment for the case of rotating concentric cylinders. Taylor himself was responsible for this, and the work continues to be a model for understanding the stability of mean flows with curved stream lines.

The advent of matched asymptotic expansions and singular perturbation analysis brought new vigor to the theory. Lin (1944, 1945) made use of these tools and re did all previous calculations, thereby confirming the earlier results that had been obtained by less sophisticated means. Experiments also gained momentum with the work of Schubauer & Skramstad (1943) in the investigation of the flat-plate boundary layer setting the standard. Here, a vibrating ribbon was employed to simulate a controlled disturbance, that is a Tollmien–Schlichting wave, at the boundary. This method is still employed by many today. Theoretical calculations were confirmed and, equally important, for the first time it became apparent that the value of the critical Reynolds number meant the stability boundary for the onset of unstable Tollmien–Schlichting waves and not the threshold for the onset of turbulence. Figure 1.6, depicting the results of this experiment, is a hallmark in this field. This conclusion has been further substantiated today. For example, Schubauer & Klebanoff (1955, 1956), Klebanoff, Tidstrom & Sargent (1962) and Gaster & Grant (1975) performed even more extensive experiments for the boundary layer.

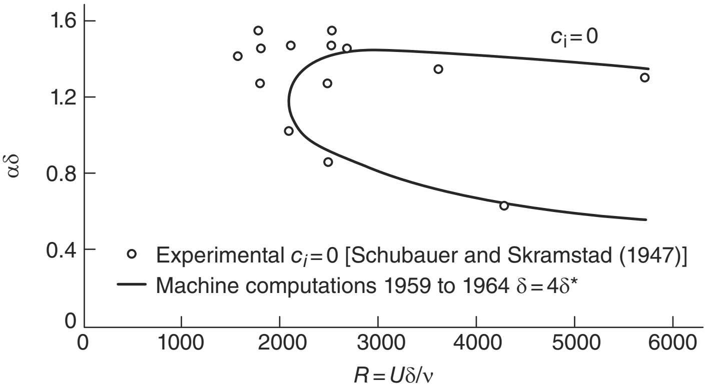

Figure 1.6 Experimental and theoretical stability results for neutral oscillations of the Blasius boundary layer (after Betchov & Criminale, 1967).

Investigating the stability of compressible flows was not done until much later with the theoretical work of Landau (1944), Lees (1947) and Dunn & Lin (1955) being the principal contributors at this time. Physically and mathematically, this is a far more complex problem and, in view of the time span it took to resolve the theory in an incompressible medium, this was understandable. A wide range of problems have been investigated here, including different prototypes and Mach numbers up to hypersonic in value. Likewise, there are experiments that have been done for these flows (see Kendall, 1966).

The use of numerical computation for stability calculations was made with the work of Brown (1959, 1961a,b, 1962, 1965), Mack (1960, 1965a,b) and Kaplan (1964) being the principal contributions. Neutral curves that were previously obtained by asymptotic theory and hand calculations are now routinely determined by numerical treatment of the governing stability equations. Such numerical evaluation has proven to be more efficient and far more accurate than any of the methods employed heretofore. Furthermore, the complete and unsteady nonlinear Navier–Stokes equations are evaluated by the use of high-order numerical methods in tandem with machines that range from the personal computer to supercomputers and the parallel class of machines, which are the standard tool for solving fluid mechanics problems today. By numerical calculations, one of the earliest results for the full Navier–Stokes calculations was obtained by Fromm & Harlow (1963), where the problem of vortex shedding from a vertical flat plate was investigated. Since this time, the complete Navier–Stokes equations are routinely used to study the vortex shedding process. Among others, Lecointe & Piquet (1984), Karniadakis & Triantafyllou (1989) and Mittal & Balachandar (1995), for example, have all numerically solved the full equations in order to investigate instability and vortex shedding from cylinders. A summary of this vortex shedding problem is provided in a review by Williamson (1996).

Effort has been made to assess nonlinearity in stability theory. Meksyn & Stuart (1951), Benney (1961, 1964) and Eckhaus (1962a,b, 1963, 1965) were all early contributors to what is now known as weakly nonlinear theory. Each effort was directed to different aspects of the problems. For example, the nonlinear critical layer, development of longitudinal or streamwise vortices in the boundary layer and the possibility of a limiting amplitude for an amplifying disturbance were examined. The role of streamwise vorticity in the breakdown from laminar to turbulent flow has recently been explored using the complete Navier–Stokes equations. For this purpose, Fasel (1990), Fasel & Thumm (1991), Schmid & Henningson (1992a,b) and Joslin, Streett & Chang (1993) have introduced oblique wave pairs at amplitudes ranging from very small to finite values. The interaction of such oblique waves leads to dominant streamwise vortex structure. When the waves have small amplitudes, the disturbances first amplify but then decay at some further downstream location. When finite, the nonlinear interactions of the vortex and the oblique waves result in breakdown.

Since the experimental setting for probing in this field is almost unequivocally one where any disturbance changes in space and only oscillates in time, thought has been given to the question of spatial instability so that theory may be more compatible with experimental data. The problem can be posed in very much the same way as the temporal one, but the equations must be adapted for this purpose (e.g., see Section 1.8 for the discussion based on Gaster (1965a,b)). This is true even if the problem is governed by the linear equations. Direct numerical simulation also has major complexities when computations are made in this way. Nevertheless, this is done. For this purpose, reference to the summaries of Kleiser & Zang (1991) and Liu (1998) can be made where the use of direct numerical simulation for many instability problems has been given. More specifically, among this vast group, Wray & Hussaini (1984) and Spalart & Yang (1987) both investigated the breakdown of the flat-plate boundary layer by use of a temporal numerical code. In other words, an initial value problem was prescribed at time $t = 0$, and the disturbance developed for later times. By contrast, when a spatial code is employed, and initial values are given at a fixed location and then the development thereafter downstream, the work of Fasel (1976), Murdock (1977), Spalart (1989), Kloker & Fasel (1990), Rai & Moin (1991a,b) and Joslin, Streett & Chang (1992, 1993) should be noted. For three-dimensional mean flows, where cross flow disturbances are present, Spalart (1990), Joslin & Streett (1994) and Joslin (1995a) studied the breakdown process by means of direct numerical simulation.

Stability theory uses perturbation analysis in order to test whether or not the equilibrium flow is unstable. Consider the flows that are incompressible, time independent and parallel or almost parallel by defining the mean state as

$$\underline{U} = (U(y), 0, 0); \quad P$$

in Cartesian coordinates where $U(y)$ is in the x-direction with y the coordinate that defines the variation of the mean flow, z is in the transverse direction and P is the mean pressure. For some flows, such as that of channel flow, this result is exact; for the case of the boundary layer or one of the free shear flows, then this is only approximate but, as already mentioned, the U-component of the velocity, $U \gg V$ and $U \gg W$, as well as U varying only weakly with x, and hence the designation of almost parallel flow. In this configuration, both x and z range from minus to plus infinity with y giving the location of the solid boundaries, if there are any. P is the mean pressure and the density is taken as constant.

Now assume that there are disturbances to this flow that are fully three-dimensional and hence

$$\underline{u} = (U(y) + \tilde{u}, \tilde{v}, \tilde{w}); \quad p = P + \tilde{p}$$

can be written for the velocity and pressure of the instantaneous flow. By assuming that the products of the amplitudes (defined nondimensionally with the measure in terms of the mean flow) of the perturbations, as well as the products of the perturbations with the spatial derivatives of the perturbations, are small, then, by subtracting the mean value terms from the combined flow, a set of linear equations can be found and are dimensionally

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} = 0, \tag{1.1}$$

for incompressibility, and

$$\frac{\partial \tilde{u}}{\partial t} + U\frac{\partial \tilde{u}}{\partial x} + \frac{dU}{dy}\tilde{v} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x} + \nu\nabla^2\tilde{u}, \tag{1.2}$$