Introduction

> In some matters you will find more polish here; in others, more plainness. But the delights of mathematics are as deeply felt here as in every other chapter.
>
> (Anonymous)

This introductory chapter lays out some fundamental notions and constructions as well as notation that will be used in subsequent chapters, often without a specific reference to them. Of necessity, many details are omitted and we only give a few proofs of the results listed. We refer the inquisitive reader interested in more details regarding any of the topics mentioned here to relevant literature.

0.1 Categories and Functors

Categories are one of the relatively recent manifestations of a crucial characteristic of mathematics, namely abstraction and generalization. Just as the notion of an abstract group arises by consideration of the formal properties of one-to-one transformations of a set onto itself, so is the notion of a category obtained from the formal properties of the class of all transformations $\alpha : X \longrightarrow Y$, of any set into another, or continuous transformations of one topological space into another, or homomorphisms of one group into another, etc.

A *category* $\mathcal{A}$ consists of a *class of objects* Obj $\mathcal{A}$ and a *class of morphisms* (or *arrows*) Mor $\mathcal{A}$ of the form $f : A_1 \longrightarrow A_2$ where the *domain* A_1 and *codomain* A_2 are in Obj $\mathcal{A}$. In addition, this class of morphisms contains *identity arrows* $1_A = id_A : A \longrightarrow A$, for every $A \in \mathcal{A}$, and there is a *composition operation* $f \circ g : A_1 \longrightarrow A_3$ (or simply fg) between morphisms of the form $g : A_1 \longrightarrow A_2$ and $f : A_2 \longrightarrow A_3$ (the codomain of g must equal the domain of f and the composition inherits the domain from g and the codomain from f). This composition satisfies the following axioms:

(a) Neutral element: $id \circ f = f$ and $f \circ id = f$.

(b) Associativity: $f \circ (g \circ h) = (f \circ g) \circ h$ for all morphisms that make these compositions possible; that is to say that $(\text{Mor } \mathcal{A}, \circ)$ is a semigroup with an identity (i.e. it is a *monoid*).

We will not concern ourselves in this work with foundational problems of category theory that stem from considerations regarding sets and classes. There is ample literature on the foundations of category theory; for a starter one can consult Mac Lane (1971a,b). A category is a *small category* if both classes Obj $\mathcal{A}$ and Mor $\mathcal{A}$ are sets that are members of a fixed universe (which is a set); it is axiomatized with Zermelo–Fraenkel axioms (for instance) in a way so as to enable construction of most of ordinary mathematics. If both Obj $\mathcal{A}$ and Mor $\mathcal{A}$ are (proper) classes, then $\mathcal{A}$ is called a *large category*. It is well known that sets are classes, but there are classes that are not sets. Mathematics of classes is often axiomatized by the so-called von Neumann–Gödel–Bernays axioms.

Given a category $\mathcal{A}$, we can form the *opposite category* $\mathcal{A}^{op}$ (some use the term *dual category*), which has the same objects as $\mathcal{A}$ and the reversed morphisms $f^{op} : A_2 \longrightarrow A_1$ that are in one-to one correspondence with the morphisms $f : A_1 \longrightarrow A_2 \in \text{Mor } \mathcal{A}$. The composition is defined in an appropriate fashion: $f^{op} g^{op} = (gf)^{op}$. It is clear that the double dual is the original: $(\mathcal{A}^{op})^{op} = \mathcal{A}$.

If $\mathcal{A}$ is a category, we will denote by $\text{Hom}_{\mathcal{A}}(A, B)$ the totality of all morphisms (arrows) $A \longrightarrow B$ in Mor $\mathcal{A}$ and call it vaguely the *Hom-set*. It is a set, if $\mathcal{A}$ is a small category. $\mathcal{B}$ is a *subcategory* of category $\mathcal{A}$, if Obj $\mathcal{B}$ is a subclass (subset) of Obj $\mathcal{A}$ and Mor $\mathcal{B}$ is a subclass (subset) of Mor $\mathcal{A}$ and $\mathcal{B}$ is a category with respect to the same composition operation for morphisms.

Usefulness of the concept of a category is demonstrated further by introduction of functors. A *(covariant) functor* $F : \mathcal{A} \longrightarrow \mathcal{B}$ between two categories consists of a pair of functions Obj $\mathcal{A} \longrightarrow$ Obj $\mathcal{B}$, $A \mapsto F(A)$, and Mor $\mathcal{A} \longrightarrow$ Mor $\mathcal{B}$, $\alpha \mapsto F(\alpha)$, with the following properties:

(1) If $\alpha : A \longrightarrow A_1 \in \text{Mor } \mathcal{A}$, then $F(\alpha) : F(A) \longrightarrow F(A_1) \in \text{Mor } \mathcal{B}$,
(2) $F(1_A) = 1_{F(A)}$,
(3) $F(\alpha\beta) = F(\alpha)F(\beta)$, whenever $\alpha\beta$ is defined.

Condition (1) means that, for every $A, A_1 \in \text{Obj } \mathcal{A}$, the functor F defines a map $\text{Hom}_{\mathcal{A}}(A, A_1) \longrightarrow \text{Hom}_{\mathcal{B}}(F(A), F(A_1))$. We say that F is a *faithful functor* if this map is injective; it is a *full functor* if the map is surjective. The *identity functor* $1_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ of $\mathcal{A}$ leaves every object and every morphism intact; this is clearly a covariant functor. Two functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{C}$ can be *composed* in a natural way to give the functor $GF : \mathcal{A} \longrightarrow \mathcal{C}$. Thus we may form a *category* **Cat** *of categories*; its objects are categories and its morphisms are functors between those categories. If $\mathcal{B}$ is a subcategory of $\mathcal{A}$, then the obvious *inclusion functor* $In : \mathcal{B} \longrightarrow \mathcal{A}$ is automatically faithful. If this functor is also full, then we say that $\mathcal{B}$ is a *full subcategory* of $\mathcal{A}$.

A *contravariant functor* $F : \mathcal{A} \longrightarrow \mathcal{B}$ between two categories consists of a pair of functions Obj $\mathcal{A} \longrightarrow$ Obj $\mathcal{B}$, $A \mapsto F(A)$, and Mor $\mathcal{A} \longrightarrow$ Mor $\mathcal{B}$, $\alpha \mapsto F(\alpha)$, with the following properties:

(1) if $\alpha : A \longrightarrow A_1 \in$ Mor $\mathcal{A}$, then $F(\alpha) : F(A_1) \longrightarrow F(A) \in$ Mor $\mathcal{B}$,
(2) $F(1_A) = 1_{F(A)}$,
(3) $F(\alpha\beta) = F(\beta)F(\alpha)$, whenever $\alpha\beta$ is defined.

A contravariant functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ may be expressed as a covariant functor $\mathcal{A}^{op} \longrightarrow \mathcal{B}$ or $\mathcal{A} \longrightarrow \mathcal{B}^{op}$.

Notation. We denote by **Sets** the category whose objects are (small) sets (members of a large enough set universe U) and whose morphisms are functions between those sets. **Grps** denotes the category of groups with morphisms being group homomorphisms. **Ab** denotes its subcategory of commutative (Abelian) groups. **Top** denotes the category of topological spaces with continuous maps as morphisms; its subcategory of compact Hausdorff spaces will be denoted by **Comp**. **Vect** denotes a category of all vector spaces over a given field or a division ring, with linear maps as its morphisms. Another example of a category is a *discrete category* $\mathcal{D}$ where the class (or a set) of morphisms consists only of the identity morphisms. For a ring R, R**Mod** denotes the category of left R-modules and **Mod**R the category of right R-modules where morphisms are module homomorphisms. In this treatise we will assume that rings have unities and that modules are unital. A functor $F : \mathcal{D} \longrightarrow R$**Mod** from a discrete category is simply a class (or a set) of R-modules. **Rings** will denote the category that has objects all rings with unities (multiplicative identities) and morphisms ring homomorphisms that preserve all the operations (binary, unary and null-ary, i.e. the additive unit element 0 and the multiplicative unit element 1). Like many categories we will mention, these examples are not small categories. Take the categories of modules R**Mod** (resp. **Mod**R) as objects and functors $F : R$**Mod** $\longrightarrow S$**Mod** (resp. $F :$ **Mod**$R \longrightarrow$ **Mod**S) as arrows to form the categories **LtMod** (respectively **RtMod**). These are large categories.

Example 0.1 *Let* $\alpha : R_1 \longrightarrow R_2 \in$ Mor **Rings** *be a non-zero ring morphism and* $M \in R_2$**Mod**. *Then we can endow* M *with an* R_1*-module structure by defining* $r_1 m =: \alpha(r_1)m$, *for* $r_1 \in R_1, m \in M$ *(multiplication on the right-hand side is in* R_2**Mod**). *Denote* M *with this* R_1*-module structure by* $\alpha_* M \in$ Obj R_1**Mod**, *and the subcategory of* R_1*-modules with this structure by* $\alpha_* R_1$**Mod**. *A morphism* $f : M_1 \longrightarrow M_2$ *in* R_2**Mod** *may be viewed in* R_1**Mod** *as the* R_1*-morphism* $\alpha_* f = f : \alpha_* M_1 \longrightarrow \alpha_* M_2$. *In this way,* α_* *is a covariant functor. On the other hand, we have a contravariant functor* $^* :$ **Rings** $\longrightarrow$ **LtMod** *(between the ring and the (left) module categories) with* $^*(R) = R$**Mod**, *and for a ring morphism* $\alpha : R_1 \longrightarrow R_2$, $^*(\alpha) : R_2$**Mod** $\longrightarrow \alpha_* R_1$**Mod** $\hookrightarrow R_1$**Mod** *is the above described functor* α_*.

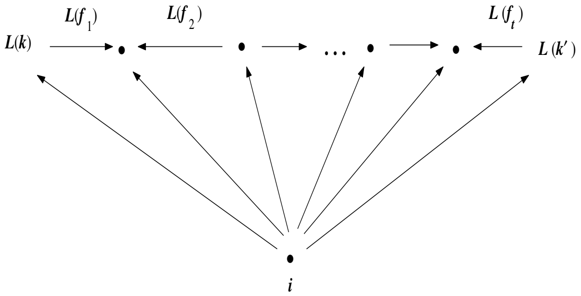
Every partially ordered set (a “poset”) $(I, \leq)$ may be viewed as a small *poset category* $\mathcal{I}$: the objects are elements of I and the morphisms $f : i \longrightarrow j$ are another notation for the inequalities $i \leq j$. Associativity of composition is ensured by transitivity of the order relation, and the identity maps exist by the reflexive property of the order. We note that, if $i \leq j$, then all arrows leading from i to j are considered to be equivalently represented by one arrow. The category that consists of such poset categories and functors (maps that preserve order) between them will be denoted by **Posets**. For a special case, if I is an *(upward) directed set* (or *udiset*), i.e. a poset such that $\forall i, j \in I \, \exists k \in I$ with $i, j \leq k$, we can likewise form this (sub)category $\mathcal{I}$. A covariant (respectively contravariant) functor $F : \mathcal{I} \longrightarrow R$**Mod** will be called a *direct* (respectively *inverse*) *system of modules*. This I-direct (respectively I-inverse) system is often denoted by $\{A_i, f_{ij}\}$, where $i, j \in I$, $A_i = F(i)$ and $f_{ij} : A_i \longrightarrow A_j = F(i \longrightarrow j)$ (respectively $f_{ij} : A_j \longrightarrow A_i = F(i \longrightarrow j)$), for $i \leq j$. The dual notion is that of a *downward directed poset* (or a *ddiset*). We will also be interested in posets that are *trees*, namely posets I such that, for every $i \in I$, the set of predecessors $(\leftarrow, i]$ is a well ordered set (see Appendix).

Following Mac Lane (1971b), a functor $L : \mathcal{K} \longrightarrow \mathcal{I}$ is called a *final functor* if:

(1) $\forall i \in \text{Obj } \mathcal{I} \; \exists k \in \text{Obj } \mathcal{K} \; \exists \, i \longrightarrow L(k) \in \text{Mor } \mathcal{I}$,

(2) each pair $i \longrightarrow L(k)$, $i \longrightarrow L(k')$ of morphisms from (1) may be connected by way of a finite number of morphisms $L(f_n)$, $n = 1, \ldots, t$, in such a way that $L(k)$ is either the domain or the codomain of $L(f_1)$, $L(k')$ is either the domain or the codomain of $L(f_t)$, and likewise for all the arrows in between,

(3) the following diagram is commutative (horizontal arrows may go in either direction):



A subcategory $\mathcal{K} \longrightarrow \mathcal{I}$ is called a *final subcategory* if the inclusion functor $In : \mathcal{K} \longrightarrow \mathcal{I}$ is final. The dual notions are that of *initial functor* and *initial subcategory*; they can also be obtained by utilizing the notions of “final” in the opposite category $\mathcal{I}^{op}$.

We will mostly apply these definitions in the context of directed sets, i.e.

CAMBRIDGE

Cambridge University Press
978-1-108-47442-9 — Slenderness
Radoslav Dimitric
Excerpt
More Information

categories of directed sets: Assume that Obj $\mathcal{I} = I$ is an upward directed set and let Mor $\mathcal{I} = \{\text{unique } f : i \longrightarrow j \mid \text{ iff } i \leq j\}$. Let $\mathcal{K}$ be a (full) subcategory of $\mathcal{I}$. Then $\mathcal{K}$ is final in $\mathcal{I}$ iff $\forall i \in I$, $\exists k \in K$ with a morphism $i \longrightarrow In(k)$, in other words $i \leq In(k)$. The second condition is automatically satisfied, since we assumed that the classes of objects are upward directed sets, so that the arrows $i \longrightarrow In(k)$ and $i \longrightarrow L(k')$ may be connected by some arrows $L(k) \longrightarrow L(k'')$ and $L(k') \longrightarrow L(k'')$. This condition is often how it is defined that a subset K is *cofinal* in an upward directed set I.

For every category $\mathcal{C}$ we have a bifunctor $\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathbf{Sets}$, defined as follows: $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is the set of arrows from A to B (we will again call it a *Hom-set*); if $f : B \longrightarrow A$, $g : C \longrightarrow D$ are morphisms of $\mathcal{C}$, then $\mathrm{Hom}_{\mathcal{C}}(f, g) : \mathrm{Hom}_{\mathcal{C}}(A, C) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(B, D)$, $\mathrm{Hom}_{\mathcal{C}}(f, g)(h) = g \circ h \circ f$. This is the *Hom-bifunctor* of the category $\mathcal{C}$. If we fix the first variable, the resulting functor $\mathrm{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \longrightarrow \mathbf{Sets}$ is covariant, and if the second variable is fixed, the resulting functor $\mathrm{Hom}_{\mathcal{C}}(-, B) : \mathcal{C} \longrightarrow \mathbf{Sets}$ is contravariant. The former Hom-functor is a *copresheaf*, the latter is *presheaf* (see Chapter 2).

Let $F, G : \mathcal{A} \longrightarrow \mathcal{B}$ be functors. A *natural transformation* $\eta : F \stackrel{\cdot}{\longrightarrow} G$ is a set of morphisms $\tau_A : F(A) \longrightarrow G(A)$ in $\mathcal{B}$, for $A \in \mathcal{A}$, with the property that, for every morphism $\alpha : A \longrightarrow A_1$ in $\mathcal{A}$, the following diagram is commutative:

$$\begin{array}{ccc} F(A) & \xrightarrow{\tau_A} & G(A) \\ {\scriptstyle F(\alpha)}\downarrow & & \downarrow{\scriptstyle G(\alpha)} \\ F(A_1) & \xrightarrow{\tau_{A_1}} & G(A_1) \end{array}$$

The natural transformation τ is a *natural equivalence* if every τ_A is an isomorphism in $\mathcal{B}$. Given categories $\mathcal{A}, \mathcal{B}$, we denote by $\mathbf{Funct}(\mathcal{A}, \mathcal{B})$ the category with functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ as objects, and morphisms the natural transformations between them. For two objects $F_1, F_2 : \mathcal{A} \longrightarrow \mathcal{B}$, we will denote $\mathbf{Nat}(F_1, F_2) = \mathrm{Hom}\,(F_1, F_2)$.

A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is an *equivalence functor* (and $\mathcal{A}$ and $\mathcal{B}$ are *equivalent categories*), if there exists another functor $G : \mathcal{B} \longrightarrow \mathcal{A}$, such that $FG \stackrel{\cdot}{\longrightarrow} 1_{\mathcal{B}}$ and $GF \stackrel{\cdot}{\longrightarrow} 1_{\mathcal{A}}$ are natural equivalences. Categories $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic categories* if FG and GF are identity functors. Clearly, isomorphic categories are equivalent, but not necessarily conversely. Note that $R\mathbf{Mod}$ and $\mathbf{Mod}R$ are isomorphic categories if R is a commutative ring, for if $\circ$ is scalar multiplication in $M \in R\mathbf{Mod}$ we can define $F : R\mathbf{Mod} \longrightarrow \mathbf{Mod}R$ by $F(M, +, \circ) = (M, +, *)$, where $m * r =: r \circ m$, $r \in R, m \in M$. We can define $G : \mathbf{Mod}R \longrightarrow R\mathbf{Mod}$ in exactly the same way where $\circ$ and $*$ have their roles switched. Define $F(f) = f$ and $G(g) = g$, for morphisms f, g in the respective categories. Then clearly FG and GF are the identity functors. In the case of non-commutative R, categories $R\mathbf{Mod}$ and $\mathbf{Mod}R$ need not be either isomorphic or equivalent.

The following result is fairly straightforward and is left to the reader to prove.

Proposition 0.2 *A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is an equivalence if and only if it is full and faithful and every object of $\mathcal{B}$ is isomorphic to an object of the form $F(A)$, for some $A \in \text{Obj } \mathcal{A}$.* □

A well-known example of equivalence of categories is the *Pontrjagin duality*. Consider the *category of compact Abelian (topological) groups* **CAb** (morphisms are continuous group homomorphisms) and let $U = \mathbb{R}/\mathbb{Z}$ denote the multiplicative group of complex numbers of modulus 1 (the *circle group*); it may be considered with the topology induced from the topology of the complex plane. We have the Hom-functor $\text{Hom}_{\mathbf{CAb}}(-, U) : \mathbf{CAb} \longrightarrow \mathbf{Ab}^{op}$. $\text{Hom}_{\mathbf{CAb}}(A, U)$ is denoted by A° and is called the *character group* of A (the notation will be justified later). Topologies are discussed in Chapter 1.

Theorem 0.3 (Pontrjagin's Duality) $\text{Hom}_{\mathbf{CAb}}(-, U)$ *is an equivalence between the category of (locally) compact Abelian groups (with continuous homomorphisms) and the opposite category of the category of (discrete) Abelian groups (so all homomorphisms are automatically continuous there).*

Proof. Define topology on A° as follows (*the compact-open topology*; see Chapter 1): the fundamental system of neighborhoods of 0 consists of the elements of the form $U(C, \epsilon) = \{\chi \in A^\circ : \chi(C) \subseteq (-\epsilon, \epsilon)\}$, for all $\epsilon > 0$ and all compact subsets C of A (they are finite, since A is discrete). [ϵ is assumed to be small enough that $(-\epsilon, \epsilon)$ does not contain non-zero subgroups of $\mathbb{R}$; $\epsilon < 1$ should suffice.] □

Another example of equivalence of categories is the so-called *Gelfand–Naumark duality*. Denote by $\mathcal{C}*$ the category whose objects are all commutative C^*-algebras with identity and whose morphisms are identity and *-preserving algebra homomorphisms. Recall that **Comp** denotes the category of compact Hausdorff spaces and continuous maps.

Theorem 0.4 (The Gelfand–Naumark Duality)
$\text{Hom}_{\mathcal{C}*}(-, \mathbb{C}) : \mathcal{C}* \longrightarrow \mathbf{Comp}$ *is an equivalence of categories.*

Proof. Note that $\text{Hom}_{\mathcal{A}}(A, \mathbb{C})$ is the maximal ideal space of A. The inverse functor assigns to each $X \in \text{Obj } \mathbf{Comp}$ the algebra $C(X)$ of all complex-valued continuous functions of X ... □

Given two covariant functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$, F is said to be a *left adjoint* of G (and G is a *right adjoint* of F) if $\text{Hom}_{\mathcal{B}}(FA, B)$ and $\text{Hom}_{\mathcal{A}}(A, GB)$ are naturally equivalent $\forall\ A \in \text{Obj } \mathcal{A}$ and $B \in \text{Obj } \mathcal{B}$. For instance, if $\text{Hom}\,(A, -) : \mathbf{Ab} \longrightarrow \mathbf{Ab}$, for a given Abelian group A, then its left adjoint is the tensor product $- \otimes A$. In a dual fashion, given two contravariant functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$, F and G are said to be *adjoint on the left* if $\mathrm{Hom}_{\mathcal{B}}(FA, B)$ and $\mathrm{Hom}_{\mathcal{A}}(GB, A)$ are naturally equivalent for all $A \in \mathrm{Obj}\ \mathcal{A}$ and $B \in \mathrm{Obj}\ \mathcal{B}$. On the other hand, these functors are said to be *adjoint on the right*, if $\mathrm{Hom}_{\mathcal{A}}(B, FA)$ is naturally equivalent to $\mathrm{Hom}_{\mathcal{B}}(A, GB)$. For instance, the functor $\mathrm{Hom}\,(-, A) : \mathbf{Ab} \longrightarrow \mathbf{Ab}$ is its own adjoint on the right, whereas the *forgetful functor* $U : \mathbf{Rings} \longrightarrow \mathbf{Sets}$ (image of every ring is its underlying set, without the operations) has the left adjoint $F : \mathbf{Sets} \longrightarrow \mathbf{Rings}$ which assigns to every set X the free ring generated by X.

Let a category $\mathcal{A}$ have small Hom-sets and let $F : \mathcal{A} \longrightarrow \mathbf{Sets}$ be a (covariant) functor. Then for $A \in \mathrm{Obj}\ \mathcal{A}$, an ordered pair (A, τ) is a *representation of* F if $\tau : \mathrm{Hom}_{\mathcal{A}}(A, -) \cong F$ is a natural equivalence. F is said to be *representable* if such a representation exists. A is called the *representing object*. Likewise, a contravariant functor $F : \mathcal{A} \longrightarrow \mathbf{Sets}$ is *representable* if there is an object $A \in \mathrm{Obj}\ \mathcal{A}$, such that F is naturally equivalent to $\mathrm{Hom}_{\mathcal{A}}(-, A)$. By this definition, the covariant functor $\mathrm{Hom}_{\mathcal{A}}(A, -)$ (resp. the contravariant functor $\mathrm{Hom}_{\mathcal{A}}(-, A)$) is a representable functor. Virtually all categorical properties are preserved under representations.

The following result is both important and well known. One of its consequences is embedding of any category into a category of functors from that category into the category of sets. Given categories $\mathcal{A}$ and $\mathcal{B} = \mathbf{Sets}$ or $\mathcal{B} = \mathbf{Ab}$, let $A \in \mathrm{Obj}\ \mathcal{A}, F \in \mathbf{Funct}(\mathcal{A}, \mathcal{B})$. If $\tau \in \mathbf{Nat}(\mathrm{Hom}_{\mathcal{A}}(A, -), F)$, then $\tau_A \in \mathrm{Hom}_{\mathcal{B}}(\mathrm{Hom}_{\mathcal{A}}(A, A), F(A))$; if we evaluate at $1_A \in \mathrm{Hom}_{\mathcal{A}}(A, A)$, then $\tau_A(1_A) \in F(A)$. The *Yoneda function* $y : \mathbf{Nat}(\mathrm{Hom}_{\mathcal{A}}(A, -), F) \longrightarrow F(A)$ is defined by $y(\tau) = \tau_A(1_A)$. Next, consider a bifunctor $N : \mathcal{A} \times \mathbf{Funct}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathcal{B}$ obtained as the composition $\mathcal{A} \times \mathbf{Funct}(\mathcal{A}, \mathcal{B}) \stackrel{\mathrm{Hom}_{\mathcal{A}} \times 1}{\longrightarrow} \mathbf{Funct}(\mathcal{A}, \mathcal{B}) \times \mathbf{Funct}(\mathcal{A}, \mathcal{B}) \stackrel{\mathrm{Hom}}{\longrightarrow} \mathcal{B}$; it is clearly defined on objects as $N(A, F) = \mathbf{Nat}(\mathrm{Hom}_{\mathcal{A}}(A, -), F)$. Likewise, we can define an evaluation bifunctor $E : \mathcal{A} \times \mathbf{Funct}(\mathcal{A}, \mathcal{B}) \longrightarrow \mathcal{B}$ defined on objects as $E(A, F) = F(A)$ and on arrows, componentwise: $E(A, F_1 \stackrel{\tau}{\longrightarrow} F_2) = F_1(A) \stackrel{\tau_A}{\longrightarrow} F_2(A)$, $E(A_1 \stackrel{\alpha}{\longrightarrow} A_2, F) = F(A_1) \stackrel{F(\alpha)}{\longrightarrow} F(A_2)$.

Lemma 0.5 (Yoneda's Lemma) *Let $\mathcal{A}$ be a category with small Hom-sets, and $\mathcal{B} = \mathbf{Sets}$ or $\mathcal{B} = \mathbf{Ab}$. Then,*

(1) *the Yoneda functions $y : \mathbf{Nat}(\mathrm{Hom}_{\mathcal{A}}(A, -), F) \longrightarrow F(A)$, defined by $y(\tau) = \tau_A(1_A)$, define a natural transformation $y : N \stackrel{\cdot}{\longrightarrow} E$,*
(2) *furthermore, $y : N \stackrel{\cdot}{\longrightarrow} E$ is a natural equivalence, namely given $A \in \mathrm{Obj}\ \mathcal{A}$ and a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$, there is a bijection*

$$\mathbf{Nat}(\mathrm{Hom}_{\mathcal{A}}(A, -), F) \cong F(A),$$

which sends each natural transformation $\tau : \mathrm{Hom}_{\mathcal{A}}(A, -) \stackrel{\cdot}{\longrightarrow} F$ to the image $\tau_A 1_A$ of the identity $A \longrightarrow A$.

One consequence is that every natural transformation of the form $\mathrm{Hom}_{\mathcal{A}}(A,-) \stackrel{\cdot}{\longrightarrow} \mathrm{Hom}_{\mathcal{A}}(B,-)$, $A, B \in \mathrm{Obj}\ \mathcal{A}$, *is of the form* $\mathrm{Hom}_{\mathcal{A}}(f,-)$, *for a unique morphism* $f : B \longrightarrow A$.

Proof. We only prove that y is a natural equivalence, for the case $\mathcal{B} = \mathbf{Ab}$ and leave the rest to the reader as an exercise. First, y is one-to-one: Let $\tau : \mathrm{Hom}_{\mathcal{A}}(A,-) \stackrel{\cdot}{\longrightarrow} F$ and let $y(\tau) = \tau_A(1_A) = 0$. We need to show $\tau = 0$. For an arbitrary $A' \in \mathrm{Obj}\ \mathcal{A}$, $\alpha \in \mathrm{Hom}_{\mathcal{A}}(A, A')$, drawing appropriate commutative diagrams will show that $\tau_{A'}(\alpha) = F(\alpha)(\tau_A(1_A))$. Thus, by the assumption, $\tau_{A'}(\alpha) = 0$, for every A', hence $\tau = 0$.

In order to show that y is onto, let $b \in F(A)$. For every $A' \in \mathrm{Obj}\ \mathcal{A}$, define the function $\tau_{A'} : \mathrm{Hom}_{\mathcal{A}}(A, A') \longrightarrow F(A')$, via $\tau_{A'}(\alpha) = (F(\alpha))(b)$, for an $\alpha \in \mathrm{Hom}_{\mathcal{A}}(A, A')$. We can show that F is additive and thus $\tau_{A'}$ is a group morphism. Now the collection $\tau_{A'}$, $A' \in \mathrm{Obj}\ \mathcal{A}$, defines a natural transformation τ, such that $y(\tau) = b$. To show that τ is natural, we need to show that, for every $\alpha : A_1 \longrightarrow A_2 \in \mathrm{Mor}\ \mathcal{A}$, we have a commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{A}}(A, A_1) & \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(A,\alpha)} & \mathrm{Hom}_{\mathcal{A}}(A, A_2) \\
{\scriptstyle \tau_{A_1}}\downarrow & & \downarrow{\scriptstyle \tau_{A_2}} \\
F(A_1) & \xrightarrow{F(\alpha)} & F(A_2)
\end{array}$$

To show commutativity, if $\gamma \in \mathrm{Hom}_{\mathcal{A}}(A, A_1)$, starting from the top left corner, clockwise compositions give us $\gamma \to \alpha\gamma \to \tau_{A_2}(\alpha\gamma) = (F(\alpha\gamma))(b)$, whereas $\gamma \to \tau_{A_1}(\gamma) \to (F(\alpha))(\tau_{A_1}(\gamma)) = F(\alpha)(F(\gamma)(b))$ are the results of counterclockwise compositions. The equality of the two compositions is established by the fact that F is a functor, and consequently $F(\alpha\gamma) = F(\alpha)F(\gamma)$. □

Let us also introduce some terminology regarding morphisms. A morphism α is *right cancellable*, or an *epic* morphism if, for all morphisms β, γ, the equality $\beta\alpha = \gamma\alpha$ implies $\beta = \gamma$. The *left cancellable* morphism is defined analogously; it is called a *monic* morphism. A morphism θ is called an *equivalence* if it has a two-sided inverse η with $\eta\theta = \theta\eta = 1$. Morphisms $\alpha : A \longrightarrow C$ and $\beta : B \longrightarrow C$ are said to be *right equivalent* if there is an equivalence $\theta : A \longrightarrow B$ with $\beta\theta = \alpha$. The right equivalence class of a monic $\alpha : C \longrightarrow A$ is called a *subobject* of A. The left equivalence class of an epic morphism $\beta : B \longrightarrow C$ is called a *quotient object* of B.

A category $\mathcal{C}$ has a *zero (null) object* 0 if $\forall X \in \mathrm{Obj}\ \mathcal{C}$, there is only one morphism $X \longrightarrow 0$ as well as a unique $0 \longrightarrow X$. A morphism $\alpha : B \longrightarrow C$ is a *monomorphism* if, for every morphism $\chi : X \longrightarrow B$, the equality $\alpha\chi = 0$ implies $\chi = 0$. Dually, $\alpha : B \longrightarrow C$ is an *epimorphism* if, for every $\chi : C \longrightarrow X$, the equality $\chi\alpha = 0$ implies $\chi = 0$. A monic need not be a monomorphism and an epic need not be an epimorphism. If $\alpha\beta$ is monic, then β must be monic; if $\alpha\beta$ is epic, then α is epic.

A *kernel* of a morphism $\alpha : B \longrightarrow C$ (denoted by $\ker\alpha$) is a morphism $\kappa : K \longrightarrow B$ with the property that $\alpha\kappa = 0$ uniquely, namely if $\chi : X \longrightarrow B$ satisfies $\alpha\chi = 0$, then there exists a unique morphism $\gamma : X \longrightarrow K$ and $\chi = \kappa\gamma$. Because of this uniqueness, we denote $K = \mathrm{Ker}\,\alpha$ (the *Kernel* of α). Dually, a *cokernel* of a morphism $\alpha : C \longrightarrow B$ (denoted by $\mathrm{coker}\,\alpha$) is a morphism $\gamma : B \longrightarrow L$ with the property that $\gamma\alpha = 0$ uniquely, i.e. if some $\chi : B \longrightarrow X$ satisfies $\chi\alpha = 0$, then there is a unique $\delta : L \longrightarrow X$ with $\chi = \delta\gamma$. Because of uniqueness, we define $\mathrm{Coker}\,\alpha = L$. The *image* of α is defined to be $\mathrm{Im}\,\alpha = \mathrm{Ker}\,(\mathrm{coker}\,\alpha)$ and the *coimage* is $\mathrm{Coim}\,\alpha = \mathrm{Coker}\,(\ker\alpha)$. In module categories, $\mathrm{Ker}\,\alpha$ and $\mathrm{Im}\,\alpha$ have the usual interpretations with $\mathrm{Im}\,\alpha \cong C/\mathrm{Ker}\,\alpha$ and $\mathrm{coker}\,(\alpha : C \longrightarrow B)$ is the usual *quotient morphism* $q : B \longrightarrow B/\mathrm{Im}\,\alpha$, and $\mathrm{Coker}\,\alpha \cong B/\mathrm{Im}\,\alpha$.

A sequence of morphisms $\cdots \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \cdots$ is *exact* at the link B, if $\mathrm{Im}\,\alpha = \mathrm{Ker}\beta$. It is an *exact sequence* (or *es*) if it is exact at every link. The term *short exact sequence* or *ses* will be reserved for any (but most frequently for the first) of the following exact sequences: $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ (for typographical reasons, this short exact sequence is sometimes denoted by $E[\alpha,\beta]$). Other short exact sequences are: $0 \longrightarrow A \longrightarrow B \longrightarrow C$ (*left exact sequence*), $A \longrightarrow B \longrightarrow C \longrightarrow 0$ (*right exact sequence*), $A \longrightarrow B \longrightarrow C$ (*middle exact sequence*), $0 \longrightarrow A \longrightarrow B$ (is the map monic?), $B \longrightarrow C \longrightarrow 0$ (is the map epic?).

Our primary interest is in categories that have arbitrary products and coproducts that come with their natural *coproduct injections* p_i and *product projections* π_i, respectively:

$$p_i : A_i \longrightarrow \coprod_{i\in I} A_i \quad \text{and} \quad \pi_i : \prod_{i\in I} A_i \longrightarrow A_i \ .$$

p_i and π_i will depend on the concrete objects. The universal properties of the coproduct (or the product) can be formulated as follows: if $q_i : A_i \longrightarrow C$ (respectively $r_i : C \longrightarrow A_i$) are morphisms, then there is a unique morphism $\coprod q_i = f : \coprod A_i \longrightarrow C$, called the *coproduct morphism* (respectively $\prod r_i = f : C \longrightarrow \prod A_i$, called the *product morphism*) with $fp_i = q_i$ (resp. $\pi_i f = r_i$), for all $i \in I$. These universal properties may also be expressed in the form of the following isomorphisms, natural in C:

$$\mathrm{Hom}_{\mathcal{C}}\left(\coprod_{i\in I} A_i, C\right) \cong \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(A_i, C);\ f \mapsto (fp_i)_{i\in I}, (f_i)_{i\in I} \mapsto \coprod f_i\,, \quad (0.1)$$

$$\mathrm{Hom}_{\mathcal{C}}\left(C, \prod_{i\in I} A_i\right) \cong \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(C, A_i);\ f \mapsto (\pi_i f)_{i\in I}, (f_i)_{i\in I} \mapsto \prod f_i\,. \quad (0.2)$$

Products and coproducts of the same set of objects coincide (they are isomorphic) if the index set is finite. If $A_i = A$ for all $i \in I$, then we abbreviate: $\prod_{i \in I} A_i = A^I$ (a *power* of A) and $\coprod_{i \in I} A_i = A^{(I)}$ (a *copower* of A).

We will sometimes abuse notation and also write $p_i : A_i \longrightarrow \prod A_i$ and $\pi_i : \coprod A_i \longrightarrow A_i$, when we really mean up_i and $\pi_i u$ respectively, where $u : \coprod_{i \in I} A_i \longrightarrow \prod_{i \in I} A_i$ is the *coproduct-to-product morphism* (see Section 0.3).

0.2 Abelian Categories and Some Categorical Constructions

A category $\mathcal{C}$ is a *preadditive category* (or by Mac Lane: an *Ab-category*) if each set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an Abelian group and the composition maps $(f, g) \mapsto f \circ g$, $\mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C)$ are bilinear with respect to the group operation. A preadditive category with a zero object 0 and finite coproducts is called an *additive category*.

$\mathcal{C}$ is an *Abelian category* if

(1) $\mathcal{C}$ is preadditive,
(2) there are finite products (and coproducts),
(3) every morphism has a kernel (and a cokernel),
(4) every monic is a kernel (and every epic is a cokernel).

Module categories are Abelian, whereas the category of groups **Grps** and **Rings** are not Abelian categories. Note that $\mathcal{C}$ is an Abelian category iff $\mathcal{C}^{op}$ is an Abelian category.

Given $f : A \longrightarrow B$, then $\mathrm{Im}\, f = \mathrm{Ker}\,(\mathrm{coker}\, f)$. In an Abelian category, $\mathrm{Ker}\,(\mathrm{coker}\, f) \cong \mathrm{Coker}\,(\ker f)$; thus the notion of *Coimage* coincides with its dual, namely that of Image. We have $\mathrm{Coim}\, f = A/\mathrm{Ker}\, f$, $\mathrm{Coker}\, f = B/\mathrm{Im}\, f$. Hence, we have the following exact sequences $0 \longrightarrow \mathrm{Ker}\, f \longrightarrow A \longrightarrow \mathrm{Coim}\, f \longrightarrow 0$ and $0 \longrightarrow \mathrm{Im}\, f \longrightarrow B \longrightarrow \mathrm{Coker}\, f \longrightarrow 0$. Therefore $\mathrm{Coim}\, f \cong \mathrm{Im}\, f$ and the following long exact sequence $0 \longrightarrow \mathrm{Ker}\, f \longrightarrow A \longrightarrow B \longrightarrow \mathrm{Coker}\, f \longrightarrow 0$ is in place. In fact the latter isomorphism may replace condition (4) in the above definition of an Abelian category. That condition may also be replaced by the following: Every morphism α has a factorization $\alpha = \gamma\beta$, where β is a cokernel and γ is a kernel.

In an Abelian category, the notions of a monic and a monomorphism, as well as epic and an epimorphism, coincide. Many of the familiar facts from module theory hold in any Abelian category. We list a couple of them here:

(1) A map that is both a monomorphism and an epimorphism is an *isomorphism* (*equivalence*).
(2) Every pair of subobjects M, N of A has the greatest lower bound; it is their *intersection*, denoted by $M \cap N$. Every pair of subobjects has the least upper bound, called the *sum* and denoted by $M + N$ or $M \cup N$. Hence the family of subobjects of any object is a lattice.