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Mathematical Constants II

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Preface

One reviewer for the first volume of *Mathematical Constants* described the book as “excellent bedtime reading” [1]. My aim here is similar to before: to gather far-flung ideas in one place, focusing on highly concrete, eminently computable results. These essays recount stories that are both successful (with depth of understanding) and tangible (in terms of numerical precision). Much mathematical research these days is necessarily abstract and qualitative, due to the enormous difficulty of the issues under consideration. Here I direct the spotlight to those rare cases when quantitative exactness still appears to be pertinent. My words from fifteen years ago (concerning purpose and scope) apply as well now as then.

A sample problem serves to illustrate my endeavor. While discussing the mass M_n of all nonisomorphic Type I inner product modules of rank n, Milnor & Husemoller [2] gave a plot of M_n on a logarithmic scale for $1 \leq n \leq 30$. They remarked that M_n is asymptotic to $C \cdot F(n)$ as $n \to \infty$, describing the function $F(n)$ exactly, but reporting only that “the constant C is approximately 0.705”. Unraveling this enigma – what is the precise nature of C? – is captivating to me. Understandably this question was incidental to the purposes of [2]; it is, however, central here to me [3]. The answer involves a quantity [4] discovered in 1860, as well as something else.

This volume is dedicated to the memory of Philippe Flajolet, a fearless leader and inspiring mentor. It is also a tribute to my parents, Charles Richard Finch and Shirley Peery Finch, and to my siblings, Valerie Jean Bridge, Gregory Charles Finch and William Robert Finch, with love and gratitude. I acknowledge a Book Fellowship from the Clay Mathematics Institute in 2004–2005, long before the magnitude of my present task became clear.

“Open the book at random”, the aforementioned reviewer wrote, evoking a few constants from many across the canvas, seedlings drawn from a vast forest. Read, learn, wander, reflect, ... “and so on into the night”.

[1] Philip J. Davis, Constants in the universe: their validation, their compilation, and their mystique, *SIAM News*, v. 37 (April 2004) n. 3.
[2] J. Milnor and D. Husemoller, *Symmetric Bilinear Forms*, Springer-Verlag, 1973, p. 50; MR0506372 (58 #22129).
[3] S. R. Finch, Minkowski-Siegel mass constants, *this volume*, §5.6.
[4] S. R. Finch, Glaisher-Kinkelin constant, *first volume*, pp. 135–145.

Notation

$\lfloor x \rfloor$	*floor function:* largest integer $\leq x$
$\lceil x \rceil$	*ceiling function:* smallest integer $\geq x$
$\{x\}$	*fractional part:* $x - \lfloor x \rfloor$
$\ln x$	*natural logarithm:* $\log_e x$
$\binom{n}{k}$	*binomial coefficient:* $\dfrac{n!}{k!(n-k)!}$
$b_0 + \dfrac{a_1 \vert}{\vert b_1} + \dfrac{a_2 \vert}{\vert b_2} + \dfrac{a_3 \vert}{\vert b_3} + \cdots$	*continued fraction:* $b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}$
$f(x) = O(g(x))$	*big O:* $\vert f(x)/g(x) \vert$ is bounded from above as $x \to x_0$
$f(x) = o(g(x))$	*little o:* $f(x)/g(x) \to 0$ as $x \to x_0$
$f(x) \sim g(x)$	*asymptotic equivalence:* $f(x)/g(x) \to 1$ as $x \to x_0$
$\sum_p$	summation over all prime numbers $p = 2, 3, 5, 7, 11, \ldots$ (only when the letter p is used)
$\prod_p$	same as $\sum_p$, with addition replaced by multiplication
$f(x)^n$	*power*: $(f(x))^n$, where n is an integer
$f^n(x)$	*iterate*: $\underbrace{f(f(\cdots f}_{n \text{ times}}(x)\cdots))$, where $n \geq 0$ is an integer
$\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt$	*error function*: same as $1 - \operatorname{erfc}(x)$

$\Phi(x) = \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{2}$ *standard normal distribution function*

${}_pF_q(\cdot;\ \cdot;\ z)$ *generalized hypergeometric function*:

$$
\begin{aligned}
&{}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) \\
&\quad = \frac{\Gamma(b_1)\Gamma(b_2)\cdots\Gamma(b_q)}{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_p)} \sum_{k=0}^{\infty} \frac{\Gamma(a_1+k)\Gamma(a_2+k)\cdots\Gamma(a_p+k)}{\Gamma(b_1+k)\Gamma(b_2+k)\cdots\Gamma(b_q+k)} \frac{z^k}{k!}
\end{aligned}
$$