1
Number Theory and Combinatorics

1.1 Bipartite, k-Colorable and k-Colored Graphs

A labeled graph G is **bipartite** if its vertex set V can be partitioned into two disjoint subsets A and B, $V = A \cup B$, such that every edge of G is of the form (a, b), where $a \in A$ and $b \in B$.

Let k be a positive integer and $K = \{1, 2, \ldots, k\}$. A labeled graph G is **k-colorable** if there exists a function $V \to K$ with the property that adjacent vertices must be colored differently. Clearly G is bipartite if and only if G is 2-colorable.

Define $c_{n,k}$ to be the number of k-colorable graphs with n vertices. We have $c_{n,1} = 1$ for $n \geq 1$ since a 1-colorable graph G cannot possess any edges. We also have $c_{1,k} = 1$ for $k \geq 1$, $c_{2,k} = 2$ for $k \geq 2$, $c_{3,2} = 7$ by Figure 1.1, $c_{3,3} = 8$, $c_{4,2} = 41$ by Figure 1.2, and $c_{4,3} = 63$. More generally, $c_{n,n-1} = 2^{n(n-1)/2} - 1$ since the total number of labeled graphs with n vertices is $2^{n(n-1)/2}$ and, of these, only the complete graph cannot be $(n-1)$-colored.

Does there exist a formula for $c_{n,k}$? The answer is yes if $k = 2$, but evidently no for $k \geq 3$. We will examine this issue momentarily, but first define a related notion.

A **k-colored graph** is a labeled k-colorable graph together with its coloring function. Let $\gamma_{n,k}$ be the number of k-colored graphs with n vertices. The point is that a k-colorable graph counts several times as a k-colored graph. Clearly $\gamma_{n,1} = 1$, $\gamma_{1,k} = k$, $\gamma_{2,2} = 6$ by Figure 1.3, $\gamma_{2,3} = 15$ by Figure 1.4, and $\gamma_{3,2} = 26$ by Figure 1.5.

When $k = 2$, the following formulas can be proved [1–3]:

$$\gamma_{n,2} = \sum_{j=0}^{n} \binom{n}{j} 2^{j(n-j)},$$

$$c_{n,2} = n! \cdot \left(\text{the } n^{\text{th}} \text{ degree Maclaurin series coefficient of } \sqrt{\Gamma(x)}\right),$$

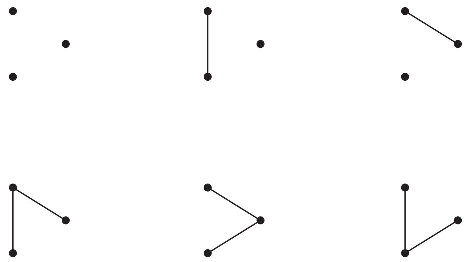

Figure 1.1 There are 7 labeled bipartite graphs with 3 vertices.

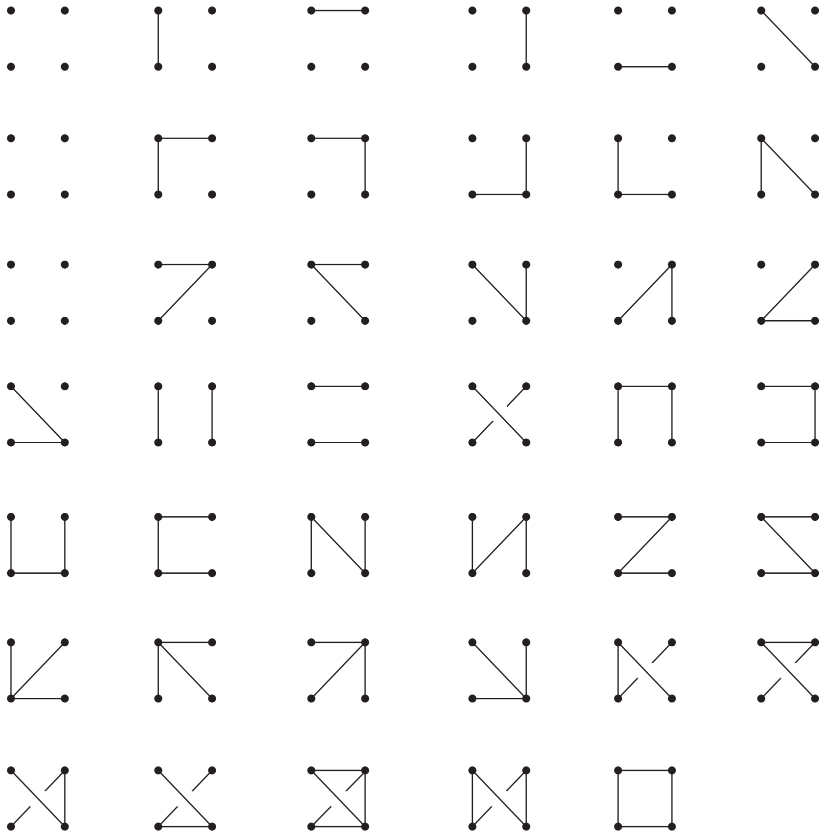

Figure 1.2 There are 41 labeled bipartite graphs with 4 vertices.

1 1   2 2   1 2   2 1   1----2   2----1

Figure 1.3 There are 6 labeled 2-colored graphs with 2 vertices.

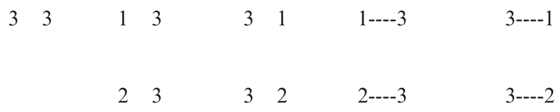

Figure 1.4 There are 15 labeled 3-colored graphs with 2 vertices (these 9 plus the preceding 6).

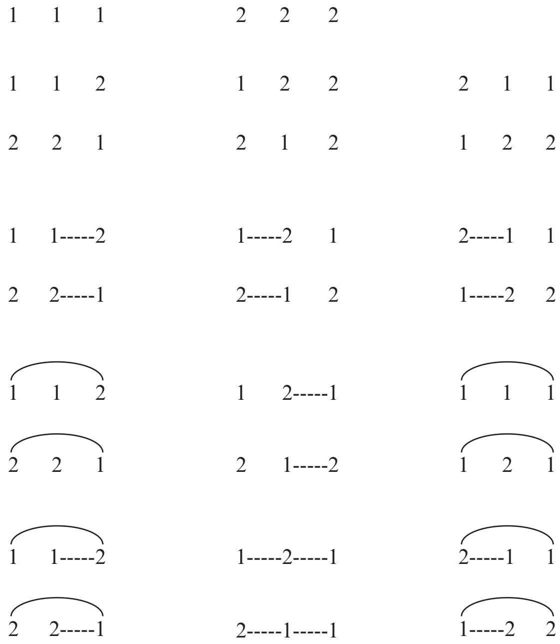

Figure 1.5 There are 26 labeled 2-colored graphs with 3 vertices.

where

$$\Gamma(x)=\sum_{i=0}^{\infty}\gamma_{i,2}\frac{x^i}{i!}.$$

For arbitrary k, we have the following recursion [4, 5]:

$$\gamma_{n,k}=\sum_{j=0}^{n}\binom{n}{j}2^{j(n-j)}\gamma_{j,k-1}$$

with initial conditions $\gamma_{0,k}=1$ and $\gamma_{n,0}=0$ for $n\geq 1$. Alternatively, we have a closed-form expression involving multinomial coefficients:

$$\gamma_{n,k}=\sum_{N}\binom{n}{n_1,n_2,\ldots,n_k}2^{\frac{1}{2}\left(n^2-n_1^2-n_2^2-\cdots-n_k^2\right)}$$

where the summation is over all nonnegative integer k-vectors $N=(n_1,n_2,\ldots,n_k)$ satisfying $n_1+n_2+\cdots+n_k=n$. There is, however, no known analogous formula for $c_{n,k}$ when $k\geq 3$.

Computations show that [4, 6]

$$\{\gamma_{n,2}\}_{n=1}^{\infty}=\{2,6,26,162,1442,18306,330626,8488962\ldots\},$$

$$\{c_{n,2}\}_{n=1}^{\infty}=\{1,2,7,41,376,5177,103237,2922446\ldots\}$$

and suggest that $\gamma_{n,2}/c_{n,2}\to 2$ as $n\to\infty$. We also have

$$\{\gamma_{n,3}\}_{n=1}^{\infty}=\{3,15,123,1635,35043,1206915,66622083,5884188675,\ldots\},$$

$$\{c_{n,3}\}_{n=1}^{\infty}=\{1,2,8,63,958,27554,\ldots\}$$

but there is insufficient data on $c_{n,3}$ to clearly suggest the asymptotic behavior of $\gamma_{n,3}/c_{n,3}$. Prömel & Steger [7], however, proved that

$$\lim_{n\to\infty}\frac{\gamma_{n,k}}{c_{n,k}}=k!$$

for each $k\geq 2$. In words, a random k-colorable graph is almost surely uniquely k-colorable (up to a permutation of colors). This is an important result since it allows us to utilize at least one term of the $\gamma_{n,k}$ asymptotics to estimate the growth of $c_{n,k}$.

We turn now to a result due to Wright [8–12]: if $n\equiv a \mod k$, where $0\leq a<k$, then

$$\gamma_{n,k}\sim C(k,a)\cdot 2^{\frac{1}{2}(1-\frac{1}{k})n^2}\cdot k^n\cdot\left(\frac{k}{\ln(2)\cdot n}\right)^{\frac{k-1}{2}}$$

as $n\to\infty$, where $C(k,a)$ is a constant that depends on n only via its residue modulo k. In fact,

$$C(k,a)=k^{\frac{1}{2}}\cdot(\ln(2))^{\frac{k-1}{2}}\cdot(2\pi)^{-\frac{k-1}{2}}\cdot L_k(a)$$

and the infinite series $L_k(a)$ will be defined for $k=2$, 3 and 4 shortly.

1.1.1 *2-Colored Graph Asymptotics*

To characterize the growth of $\gamma_{n,k}$, by the above, it is sufficient to determine $C(k,a)$ for each $0\leq a<k$. We have here

$$\begin{aligned}L_2(a)&=\sum_{r=-\infty}^{\infty}2^{-\frac{1}{2}r^2-\frac{1}{2}(a-r)^2+\frac{1}{4}a^2}\\&=\sum_{r=-\infty}^{\infty}2^{-\frac{1}{4}(a-2r)^2}=\begin{cases}2.1289368272\ldots & \text{if } a=0,\\ 2.1289312505\ldots & \text{if } a=1.\end{cases}\end{aligned}$$

These two constants also appear with regard to the asymptotic enumeration of partially ordered sets [13] and of linear subspaces of $\mathbb{F}_2^n$ [14], where $\mathbb{F}_2$ is the binary field with arithmetic modulo 2. Therefore

$$C(2,a)=\begin{cases}1.0000013097\ldots=1+\varepsilon & \text{if } a=0,\\ 0.9999986902\ldots=1-\varepsilon & \text{if } a=1\end{cases}$$

where $\varepsilon=1.3097396978\ldots\times10^{-6}$. In fact, all of the constants $C(k,a)$ we examine are close to 1; thus we shall focus on difference with 1 henceforth.

1.1.2 *3-Colored Graph Asymptotics*

We have here

$$\begin{aligned}L_3(a) &= \sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}2^{-\frac{1}{2}r^2-\frac{1}{2}s^2-\frac{1}{2}(a-r-s)^2+\frac{1}{6}a^2}\\ &= \sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}2^{-\frac{1}{3}\left(a^2-3ar+3r^2-3as+3rs+3s^2\right)}\end{aligned}$$

and therefore

$$C(3,a)=\begin{cases}1+2\varepsilon & \text{if } a=0,\\ 1-\varepsilon & \text{if } a=1 \text{ or } 2\end{cases}$$

where $\varepsilon=1.7060611047...\times10^{-8}$.

1.1.3 *4-Colored Graph Asymptotics*

All planar graphs are 4-colorable by the famous Four Color Theorem. We have here [4, 6]

$$\{\gamma_{n,4}\}_{n=1}^{\infty}=\{4,28,340,7108,254404,15531268,1613235460,284556079108,\ldots\},$$

$$\{c_{n,4}\}_{n=1}^{\infty}=\{1,2,8,64,1023,32596,\ldots\},$$

$$\begin{aligned}L_4(a) &= \sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}2^{-\frac{1}{2}r^2-\frac{1}{2}s^2-\frac{1}{2}t^2-\frac{1}{2}(a-r-s-t)^2+\frac{1}{8}a^2}\\ &= \sum_{r=-\infty}^{\infty}\sum_{s=-\infty}^{\infty}\sum_{t=-\infty}^{\infty}2^{-\frac{1}{8}\left(3a^2-8ar+8r^2-8as+8rs+8s^2-8at+8rt+8st+8t^2\right)}\end{aligned}$$

and therefore

$$C(4,a)=\begin{cases}1+\delta & \text{if } a=0,\\ 1-\varepsilon & \text{if } a=1 \text{ or } 3,\\ 1-\delta+2\varepsilon & \text{if } a=2,\end{cases}$$

where $\delta=4.2421496651\ldots\times10^{-9}$ and $\varepsilon=2.5731271141...\times10^{-12}$. A simple relationship between δ and ε is not apparent.

Higher-order asymptotics for $\gamma_{n,k}$ are possible, due to Wright [8]; the corresponding constants await study. Observe that terms beyond the first need not necessarily apply for $c_{n,k}$.

A random k-colorable graph is almost surely connected [10, 12, 15] and is almost surely k-chromatic (meaning that $k-1$ colors will not suffice to color all n vertices). The asymptotics discussed above therefore apply to these important subclasses as well.

Enumerating unlabeled k-colorable graphs (that is, non-isomorphic types of labeled k-colorable graphs) is also a difficult computational problem [16]. A general result due to Prömel [17] provides that $c_{n,k}/n!$ is the associated asymptotic formula.

[1] H. S. Wilf, *generatingfunctionology*, 2nd edn., Academic Press, 1994, pp. 86–89; MR1277813 (95a:05002).
[2] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge Univ. Press, 2009, p. 138; MR2483235 (2010h:05005).
[3] D. A. Klarner, The number of graded partially ordered sets, *J. Combin. Theory* 6 (1969) 12–19; MR0236035 (38 #4333).
[4] R. C. Read, The number of k-colored graphs on labelled nodes, *Canad. J. Math.* 12 (1960) 410–414; MR0114775 (22 #5594).
[5] F. Harary and E. M. Palmer, *Graphical Enumeration*, Academic Press, 1973, pp. 16–18; MR0357214 (50 #9682).
[6] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A000684, A000685, A000686, A047863, A047864, A084279, and A084280.
[7] H. J. Prömel and A. Steger, Random l-colorable graphs, *Random Structures Algorithms* 6 (1995) 21–37; MR1368832 (96j:05095).
[8] E. M. Wright, Counting coloured graphs, *Canad. J. Math*. 13 (1961) 683–693; MR0133252 (24 #A3086).
[9] E. M. Wright, Counting coloured graphs. II, *Canad. J. Math*. 16 (1964) 128–135; MR0177913 (31 #2171).
[10] R. C. Read and E. M. Wright, Coloured graphs: A correction and extension, *Canad. J. Math*. 22 (1970) 594–596; MR0263695 (41 #8296).
[11] E. M. Wright, Counting coloured graphs. III, *Canad. J. Math*. 24 (1972) 82–89; MR0289368 (44 #6559).
[12] F. Harary and R. W. Robinson, Labeled bipartite blocks, *Canad. J. Math*. 31 (1979) 60–68; MR0518706 (80b:05036).
[13] S. R. Finch, Lengyel's constant: Chains in the subset lattice of S, *first volume*, pp. 317–318.
[14] S. R. Finch, Transitive relations, topologies and partial orders, *this volume*, §1.2.
[15] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A001832, A084283, and A084284.
[16] N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, A033995, A076315, and A076316.
[17] H. J. Prömel, Counting unlabeled structures, *J. Combin. Theory Ser. A* 44 (1987) 83–93; MR0871390 (87m:05014).

1.2 Transitive Relations, Topologies and Partial Orders

Let S be a set with n elements. A subset R of $S \times S$ is a **binary relation** (or **relation**) on S. The number of relations on S is 2^{n^2}. Equivalently, there are 2^{n^2} labeled bipartite graphs on $2n$ vertices, assuming the bipartition is fixed and equitable.

A relation R on S is **reflexive** if for all $x \in S$, we have $(x, x) \in R$. The number of reflexive relations on S is $2^{n(n-1)}$.

A relation R on S is **antisymmetric** if for all $x, y \in S$, the conditions $(x, y) \in R$ and $(y, x) \in R$ imply that $x = y$. The number of antisymmetric relations on S is $2^n \cdot 3^{n(n-1)/2}$.

A relation R on S is **transitive** if for all $x, y, z \in S$, the conditions $(x, y) \in R$ and $(y, z) \in R$ imply that $(x, z) \in R$. There is no known general formula for the number T_n of transitive relations on S. It is surprising that such a simply-stated counting problem remains unsolved [1–6].

A **topology** on S is a collection Σ of subsets of S that satisfy the following axioms:

- $\emptyset \in \Sigma$ and $S \in \Sigma$
- the union of any two sets in Σ is in Σ
- the intersection of any two sets in Σ is in Σ.

Note that since S is finite, our phrasing of the second axiom is correct. No one knows a general formula for the number U_n of topologies on S. Also, a topology on S is a **T0 topology** if it additionally satisfies a (weak) separation axiom:

- for any pair of distinct points in S, there is a set in Σ containing one point but not the other.

Again, no one knows a general formula for the number V_n of T0 topologies [7].

A **quasi-order** on S is a relation that is both reflexive and transitive. Let Q_n denote the number of such relations. Other uses of the phrase "quasi-order" exist and so care must be taken when reviewing the literature. There is a one-to-one correspondence between the topologies on S and the quasi-orders on S; hence $Q_n = U_n$.

A **partial order** on S is a quasi-order that is antisymmetric as well. Let P_n denote the number of such relations. We usually write $x \leq y$ if $(x, y) \in R$ and, moreover, $x < y$ if $x \neq y$. There is a one-to-one correspondence between the T0 topologies on S and the partial orders on S; hence $P_n = V_n$.

Further connections between P_n and Q_n, and between P_n and T_n, can be expressed in terms of Stirling numbers of the second kind [1, 8]:

$$Q_n = \sum_{k=1}^{n} S_{n,k} P_k, \quad T_n = \sum_{k=1}^{n} \left(\sum_{j=0}^{k} \binom{n}{j} S_{n-j,k-j} \right) P_k$$

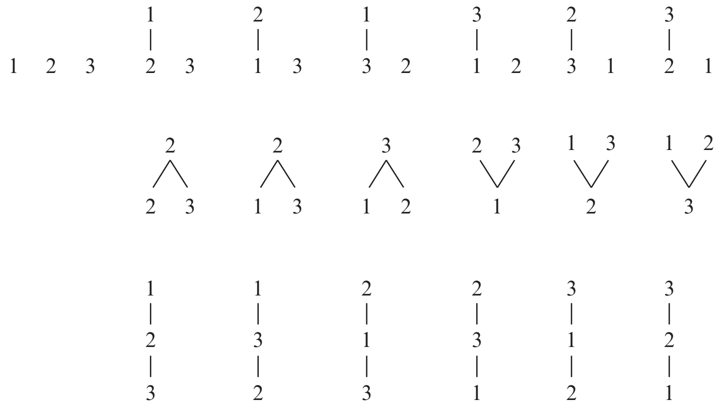

Figure 1.6 There are 19 labeled posets with 3 elements, that is, $P_3 = 19$.

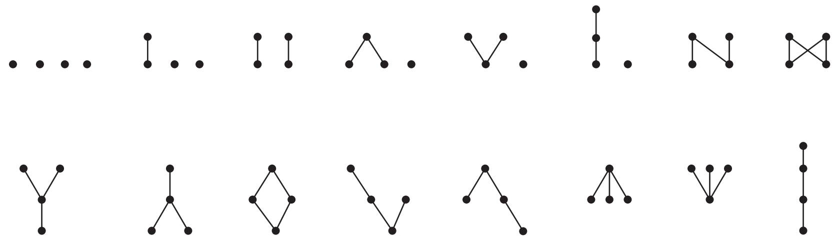

Figure 1.7 There are 16 unlabeled posets with 4 elements, that is, $p_4 = 16$.

and hence [9, 10]

$$Q_n \sim P_n, \quad T_n \sim 2^n P_n$$

as $n \to \infty$. It is therefore sufficient to focus on just one of these sequences; we choose $\{P_n\}$, which enumerates labeled posets (see Figure 1.6) as opposed to $\{p_n\}$, which enumerates unlabeled posets (see Figure 1.7). The existence of an edge (x, y) in any of the graphs pictured here indicates that $x < y$ and y is drawn above x.

Even though a closed-form expression for P_n is unknown, progress has been made in understanding the asymptotics of

$$\{P_n\}_{n=1}^{\infty} = \{1, 3, 19, 219, 4231, 130023, 6129859, 431723379, \ldots\}.$$

Kleitman & Rothschild [11] deduced that

$$\frac{\ln(P_n)}{\ln(2)} = \frac{n^2}{4} + O\left(n^{\frac{3}{2}} \ln(n)\right)$$

and later sharpened this to [12]

$$\frac{\ln(P_n)}{\ln(2)} = \frac{n^2}{4} + \frac{3n}{2} + O(\ln(n)).$$

Building on their work, several authors [10, 13–16] gave the following improvement:

$$P_n \sim C_a \cdot \sqrt{\frac{2}{\pi}} \cdot 2^{\frac{n^2}{4}+\frac{3n}{2}+\frac{1}{4}} \cdot n^{-\frac{1}{2}}$$

where $n \equiv a \bmod 2$ and $a \in \{0, 1\}$, and where

$$C_1 = \sum_{k=-\infty}^{\infty} 2^{-k^2} = 2.1289368272... = \pi \cdot (0.8058800428...) \cdot 2^{-\frac{1}{4}},$$

$$C_0 = \sum_{k=-\infty}^{\infty} 2^{-(k-\frac{1}{2})^2} = 2.1289312505... = \pi \cdot (0.8058779318...) \cdot 2^{-\frac{1}{4}}.$$

It is interesting that the constant depends on the parity of n.

The asymptotics of the unlabeled case [17, 18]:

$$\{p_n\}_{n=1}^{\infty} = \{1, 2, 5, 16, 63, 318, 2045, 16999, \ldots\}$$

turn out to satisfy

$$p_n \sim \frac{P_n}{n!} \sim C_a \cdot \frac{1}{\pi} \cdot 2^{\frac{n^2}{4}+\frac{3n}{2}+\frac{1}{4}} \cdot e^n \cdot n^{-n-1}$$

thanks to a general result due to Prömel [19].

See [20, 21] for more appearances of the constants C_0 and C_1. It's believed that, for any asymptotic enumeration problem where a typical member is based on a bipartite graph, these constants are likely to occur. Alternative representations include [16, 22]:

$$C_1 = \sqrt{\frac{\pi}{\ln(2)}} \sum_{k=-\infty}^{\infty} \exp\left(\frac{-\pi^2}{\ln(2)} k^2\right), \quad C_0 = \sqrt{\frac{\pi}{\ln(2)}} \sum_{k=-\infty}^{\infty} (-1)^k \exp\left(\frac{-\pi^2}{\ln(2)} k^2\right)$$

from which the strict inequality $C_0 < C_1$ becomes obvious.

1.2.1 *Natural Partial Orders*

Consider the set $S = \{1, 2, \ldots n\}$ equipped with the usual total ordering $\leq$. A **natural partial order** $\preceq$ on S is a partial ordering that is compatible with $\leq$ (meaning that if $x \preceq y$, then $x \leq y$). This is equivalent to saying that $(S, \leq)$ is a **linear extension** of $(S, \preceq)$. Define σ_n to be the number of natural partial orders on S, then [23–25]

$$\{\sigma_n\}_{n=1}^{\infty} = \{1, 2, 7, 40, 357, 4824, 96428, 2800472, \ldots\}$$

(see Figure 1.8).

Brightwell, Prömel & Steger [16] proved that

$$\sigma_n \sim \begin{cases} \frac{1}{2}\eta^2 \cdot C_1 \cdot 2^{\frac{n^2}{4}} \cdot n = (12.7636300229...) \cdot 2^{\frac{n^2}{4}} \cdot n & \text{if } n \text{ is even,} \\ \frac{1}{2}\eta^2 \cdot C_0 \cdot 2^{\frac{n^2}{4}} \cdot n = (12.7635965889...) \cdot 2^{\frac{n^2}{4}} \cdot n & \text{if } n \text{ is odd,} \end{cases}$$

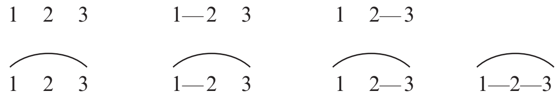

Figure 1.8 There are 7 natural partial orders on $\{1, 2, 3\}$, that is, $\sigma_3 = 7$.

where

$$\eta = \prod_{j=1}^{\infty} \left(1 - 2^{-j}\right)^{-1} = 3.4627466194\ldots$$

is a digital search tree constant [26]. These constants also arise when determining the average number λ_n of linear extensions of S, where S is a *random* poset on n points [16, 27]:

$$\lambda_n \sim \begin{cases} \dfrac{\eta^2 C_1}{2^{5/4} C_0} \cdot \left(\frac{n}{2}\right)!^2 \cdot n \cdot 2^{-n/2} = (5.0414454338\ldots) \cdot \left(\frac{n}{2}\right)!^2 \cdot n \cdot 2^{-n/2}, \\[2ex] \dfrac{\eta^2 C_0}{2^{5/4} C_1} \cdot \left(\frac{n-1}{2}\right)! \cdot \left(\frac{n+1}{2}\right)! \cdot n \cdot 2^{-n/2} = (5.0414190220\ldots) \cdot \left(\frac{n-1}{2}\right)! \\ \quad \cdot \left(\frac{n+1}{2}\right)! \cdot n \cdot 2^{-n/2} \end{cases}$$

when n is even, respectively, n is odd.

Consider instead the set S of all 2^n subsets of $\{1, 2, \ldots, n\}$ equipped with the usual partial ordering $\subseteq$. Define τ_n in a manner analogous to σ_n. We observe that $\lambda_n \cdot P_n \sim n! \cdot \sigma_n$ and wonder what the corresponding asymptotics for τ_n might be.

1.2.2 *Evolving Posets*

An interesting variation is as follows. What is the number N_ρ of partial orders on S with the property that a specified fraction ρ of the $n(n-1)/2$ pairs of distinct points are comparable? (If necessary, $\rho n(n-1)/2$ is rounded to the nearest integer.) Dhar [28, 29] investigated this question in the limit as $n \to \infty$ and proposed a lattice gas model (with infinitely many phase transitions) based on the evolution of N_ρ as ρ increases. A highly intricate analysis of Dhar's model was completed in [30–32].

[1] J. Klaska, Transitivity and partial order, *Math. Bohemica* 122 (1997) 75–82; MR1446401 (98c:05006).
[2] J. Klaska, History of the number of finite posets, *Acta Univ. Mathaei Belii Nat. Sci. Ser. Ser. Math.* 5 (1997) 73–84; MR1618881 (99k:06001).
[3] M. Erné and K. Stege, Counting finite posets and topologies, *Order* 8 (1991) 247–265; MR1154928 (93b:06004).
[4] P. Renteln, Geometrical approaches to the enumeration of finite posets: An introductory survey, *Nieuw Arch. Wisk.* 14 (1996) 349–371; MR1430049 (98g:06010).