Preface

Since ancient times, there has been a struggle between mathematics and its philosophy. As soon as there seems to be a settled view of the nature of mathematics, some new mathematical discovery comes along to disrupt it. Thus, the Pythagorean view that 'all is number' was disrupted by the discovery of irrational lengths, and the philosophy of mathematics had to expand to include a separate field of geometry. But this raised the question, Can the geometric view be reconciled with the numerical view? If so, how? And so it went, for millennia.

In many cases, advances *in* mathematics changed ideas *about* mathematics, by forcing the acceptance of concepts previously thought impossible or paradoxical. Thus mathematics disrupted philosophy. In the opposite direction, philosophy kept mathematics honest by pointing out contradictions and suggesting how concepts might be clarified in order to resolve them. Sometimes the philosopher and the mathematician were one and the same person – such as Descartes, Leibniz, or Bolzano – so one might almost say that mathematics is an especially rich and stable branch of philosophy. At any rate, if one is to understand the past and present state of the philosophy of mathematics, one must first understand mathematics, and its history.

The aim of the present Element is to give a brief introduction to mathematics and its history, with particular emphasis on events that shook up its philosophy. If you like, it is a book on 'mathematics for philosophers'. I try not to take a particular philosophical position, except to say that I believe that mathematics guides philosophy, more so than the other way round. As a corollary, I believe that mathematicians have made important contributions to philosophy, even when it was not their intention.

Each section begins with a preview of topics to be discussed and ends with a section highlighting the philosophical questions raised by the mathematics. The same themes recur from section to section – intuition and logic, meaning and existence, and the discrete and the continuous – but they evolve under the influence of new mathematical discoveries.

Experts may be surprised that there is little or no mention of philosophies of mathematics that were prominent in the twentieth century – platonism, logicism, formalism, nominalism, and intuitionism, for example. This is partly because I find none of them adequate, but mainly because I hope to look at the philosophy of mathematics without being influenced by labels. I want to present as much *philosophically instructive mathematics* as possible and leave readers to decide how it should be sorted and labelled in philosophical terms. My hope is that this Element will equip readers with a 'mathematical lens' with which to view many philosophical issues.

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1 Irrational Numbers and Geometry

PREVIEW

The source of many issues in the philosophy of mathematics – the nature of proof and truth; the meaning and existence of numbers; the role of infinity; and the relation between geometry, algebra, and arithmetic – is Euclid's *Elements* from around 300 BCE. The *Elements* is best known for its axiomatic geometry – Euclidean geometry – which includes proofs of signature results such as the Pythagorean theorem and the existence of exactly five regular polyhedra. However, the *Elements* also includes fundamental ideas of number theory, such as the existence of infinitely many prime numbers, the Euclidean algorithm for greatest common divisor, and (an equivalent of) unique prime factorization.

In Euclid's time, as now, there was a conceptual gulf between geometry and number theory – between measuring and counting, or between the continuous and the discrete. The major reason for this gulf was the existence of irrationals, discovered before Euclid's time by the Pythagoreans and, by the time of the *Elements*, the subject of a sophisticated 'theory of proportions'. This theory, in Book V of the *Elements*, made a tenuous bridge between the continuous and the discrete. The bridge was gradually strengthened over the centuries by the work of later mathematicians, but not without philosophical conflicts and mathematical surprises.

These issues are the subject of this section and the next.

1.1 The Pythagorean Theorem

The Pythagorean theorem was discovered independently several times in human history, and in several different cultures. So if any theorem typifies mathematics – and its universality – this is it. Figure 1 illustrates the theorem: the (grey) square on the hypotenuse of the (white) right-angled triangle is equal to sum of the (black) squares on the other two sides.

Figure 2 shows a plausible 'proof by picture' of the theorem: the grey square equals the big square minus four copies of the triangle, which in turn equals the sum of the two black squares.

Most of the independent discoveries of the theorem were probably like this, and indeed the human visual system has many mathematical discoveries to its credit. Nevertheless, it was the radically different *axiomatic* path to theorems,

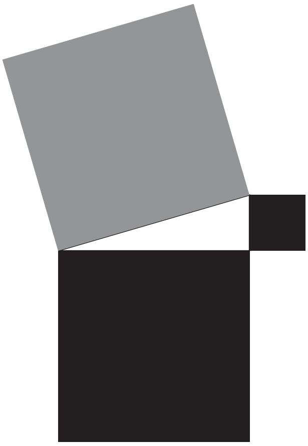

Figure 1 The Pythagorean theorem.

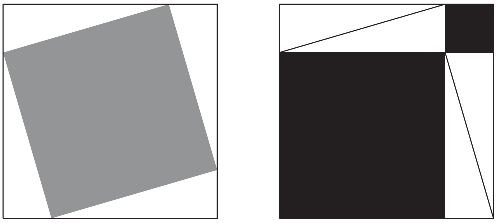

Figure 2 Seeing the Pythagorean theorem.

discussed in Section 1.4, that set the direction of mathematics for the next 2000 years.

But before the axiomatic path was established, the Pythagorean theorem provoked another important conceptual development: a distinction between length and number. Legend has it that the philosophy of the Pythagoreans was ‘all is number’, prompted by the discovery that whole number ratios govern musical harmony. This philosophy was overturned when irrational ratios were found in geometry – because of the Pythagorean theorem.

1.2 Irrationality

The Pythagorean theorem talks about sums of squares – an operation we will say more about below – but indirectly, it also tells us something about lengths. In particular, it says that if a triangle has perpendicular sides of length 1, then its hypotenuse has the length l whose square is 2. Using the modern notation l^2 to denote the square of side l, we have $l^2 = 1^2 + 1^2 = 2$.

Now (again using modern notation), suppose that l is rational, in which case we can suppose that $l = m/n$, where m and n are whole numbers. We can also suppose that m and n have no common divisor except 1, since any other common divisor could be divided out of m and n in advance, without changing l. Under these conditions we can derive a contradiction by the following series of implications (these probably go back to the Pythagoreans, but the first known hint of such a proof is in Aristotle's Prior Analytics 1.23):

$$\begin{aligned}
l = m/n &\Rightarrow 2 = m^2/n^2 && \text{(squaring both sides)}\\
&\Rightarrow 2n^2 = m^2 && \text{(multiplying both sides by } n^2\text{)}\\
&\Rightarrow m^2 \text{is even} &&\\
&\Rightarrow m \text{ is even, say, } = 2p && \text{(since the square of an odd number is odd)}\\
&\Rightarrow 2n^2 = (2p)^2 = 4p^2 && \text{(substituting } m = 2p \text{ in } 2n^2 = m^2\text{)}\\
&\Rightarrow n^2 = 2p^2 &&\\
&\Rightarrow n^2 \text{is even} &&\\
&\Rightarrow n \text{ is even} &&\\
&\Rightarrow 2 \text{ divides both } m \text{ and } n, &&\\
&\qquad \text{contrary to the assumption of no common divisor.} &&
\end{aligned}$$

Since it is contradictory to assume that l is a ratio of whole numbers, l is an irrational length. The Greeks often expressed this by saying that the side and hypotenuse of the right-angled triangle with equal sides are *incommensurable* – not whole number multiples of any common unit of measure.

In the view of the Pythagoreans, the lack of a common unit of length meant that lengths are not numbers, because 'numbers' to them were whole numbers and their ratios. In particular, sums and products of lengths are not necessarily like sums and products of numbers, so the concept of 'sum of squares' needs clarification. In the next section we will see how Euclid handled sums and products of lengths.

1.3 Operations on Lengths and Numbers

By denying that irrational lengths could be numbers, yet allowing that they could be squared and added, the Greek mathematicians after Pythagoras had to define sum and product in purely geometric terms.

The sum of two lengths is defined in the obvious way suggested by Figure 3. The lengths are represented by two line segments a and b, and $a + b$ is obtained by joining these segments end to end.

It follows easily that $a + b = b + a$ and $a + (b + c) = (a + b) + c$ (commutative and associative laws). It is also clear, since the sum of lengths is a length,

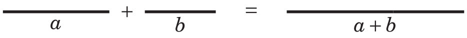

Figure 3 The sum of two lengths.

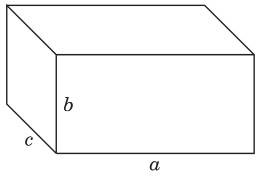

Figure 4 The product of three lengths.

that any number of lengths can be added. Thus lengths behave exactly like numbers as far as addition is concerned.

The behaviour of products is not so simple. The product of lengths a and b is not a length but the rectangle with perpendicular sides a and b. And the product of lengths a, b, and c is the rectangular box with perpendicular sides a, b, and c (Figure 4).

It is clear from these definitions that $ab = ba$ and $a(bc) = (ab)c$, and it can also be seen that $a(b + c) = ab + ac$ (the latter is actually a special case of Euclid's Proposition 1 of Book II of the *Elements*). Thus, to the extent that sum and product are defined, lengths satisfy the same laws as positive numbers. The trouble is that they are defined only to a limited extent, so the algebra of lengths is crippled. Products of more than three lengths are not admitted, because they have no geometric counterpart. Likewise, products can be added only when each is of the same 'dimension', that is, a product of the same number of lengths.

Finally, there is a complicated, though geometrically natural, notion of *equality*. It says, for example, that two rectangles R and S are equal if R can be cut into a finite number of triangles which reassemble to form S. We say more about Euclid's theory of equality for rectangles, and other polygons, in the next section. Remarkably, this theory is perfectly adequate for polygons, because any two polygons of equal area (in the modern sense) are actually equal in Euclid's sense. However, the theory is not adequate for polyhedra, as was shown by Dehn (1900). Dehn showed that a cube and regular tetrahedron of equal volume are not equal in Euclid's sense.

1.4 Axiomatics

The power of the axiomatic method is charmingly described by John Aubrey in his *Brief Lives*, speaking of Thomas Hobbes:

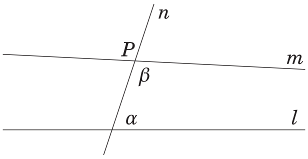

Figure 5 Non-parallel lines.

> He was 40 yeares old before he looked on geometry; which happened accidently. Being in a gentleman's library … Euclid's *Elements* lay open, and 'twas the 47 *Elements*, Book I. He read the proposition. 'By G–', sayd he, 'this is impossible!' So he reads the demonstration of it, which referred him back to such a proposition; which proposition he read. That referred him back to another, which he also read … that at last he was demonstratively convinced of that trueth. This made him in love with geometry.

Proposition 47 of Book I, incidentally, is the Pythagorean theorem. The *Elements* is the first systematic account of theorems and proofs that has come down to us, and it became the standard way of presenting mathematics in the Western world (and later the Islamic world) for the next 2000 years.

Euclid begins with a small number of basic assumptions (axioms) and deduces all theorems from them by logic. His axioms include simple statements about points, lines, length, and angle. There are also statements about equality, addition, and subtraction, such as 'things which are equal to the same thing are equal to each other' and 'if equals be added to equals then the wholes are equal'. The principles of logic are not explicitly stated. The most important axiom, needed for the Pythagorean theorem and many others, is the parallel axiom. Euclid states it as follows, in the translation by Heath (1956):

> That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

This rather long-winded statement is illustrated in Figure 5. The line n falls on the lines l and m, making angles α and β on the right with $\alpha + \beta < \pi$. The conclusion is that l and m then meet somewhere on the right. Thus the parallel axiom actually gives a condition for lines not to be parallel.

It follows (not quite obviously) that there is exactly one parallel to a given line l through a given point P outside l, namely the line m for which $\alpha + \beta = \pi$.

The complicated character of the parallel axiom provoked many attempts to eliminate it by showing that it follows from Euclid's other axioms. But all such attempts failed. This led, in the nineteenth century, to a thorough examination of the axiomatic method and to subsequent analysis of its scope and limits. We pick up this story later.

1.5 Philosophical Issues

According to legend, the Pythagoreans were the first to propose a philosophy of mathematics, in fact a very simple 'theory of everything': *all is number*. It is said that they observed the role of whole numbers in musical harmony and jumped from there to the conclusion that the whole universe is ruled by whole numbers and their ratios. The echoes of this philosophy are still heard in phrases like 'the harmony of the spheres'.

Whatever its details may have been, the Pythagorean philosophy was disrupted by the discovery of irrational quantities such as $\sqrt{2}$. Irrationals were unacceptable as numbers, but unavoidable in geometry, since no one could deny that if a square exists, then so does its diagonal. This led to the separation of number theory and geometry seen in Euclid's *Elements* but also to the theory found in the *Elements* Book V. The 'theory of proportions' found in Book V establishes a point of contact between (rational) numbers and geometric quantities, though without fully reconciling the two.

Much of the subsequent history of mathematics, and its philosophy, grows from the struggle to reconcile the concepts of number and quantity, or the discrete and the continuous, or the rational (logical) and the visual (intuitive). The development of mathematical philosophy accompanies this struggle, as we will see in the sections that follow. At the end of each section I will give a historical update, as it were, of philosophical developments, under the headings of logic and intuition, meaning and existence, and discrete and continuous. As a mathematician, I prefer to think in these terms, but I hope that philosophers will be able to translate the philosophical content of my remarks into their own preferred terms.

Intuition and logic. Probably in an attempt to work precisely with geometric quantities, Euclid's *Elements* is the first known example of the axiomatic approach to truth, whereby theorems are deduced from axioms by logic. However, his axioms are incomplete, and there are frequent appeals to intuition, one even in his first proposition. Thus the *Elements* unintentionally illustrates how hard it is to avoid unconscious assumptions in mathematical reasoning.

Meaning and existence. Euclid also undercuts what we now consider to be the axiomatic method by attempting to define primitive concepts such as 'point' and 'line'. He also restricts the concept of 'number' essentially to the natural numbers and their ratios. The irrationality of $\sqrt{2}$ is thought to disqualify it from being a number, but Euclid did not prescribe what the properties of numbers should be.

Discrete and continuous. Because of its sensitivity to irrational quantities, the *Elements* generally makes a clear separation between the concepts of number and quantity, or between the discrete and the continuous. But Book V begins a possible merger between the two, as we will see later. This illustrates the sometimes opposing tendencies of mathematics and philosophy. Mathematicians generally have the outlook expressed by Poincaré in 1908:

> I think I have already said somewhere that mathematics is the art of giving the same name to different things. It is enough that these things, though differing in matter, should be similar in form, to permit of their being, so to speak, run in the same mould. When language has been well chosen, one is astonished to find that all demonstrations made for a known object apply immediately to many new objects: nothing requires to be changed, not even the terms, since the names have become the same. (see Poincaré 1952, 34)

In other words, mathematicians consider things to be the same if they have the same behaviour. Philosophers, however, like to make distinctions: they look for reasons why things should *not* be considered the same. Sometimes a distinction seems to be justified, as was the Greek distinction between numbers and geometric quantities such as length. But mathematics seeks to erase distinctions where possible. The long evolution of the real number concept can be viewed as a project to erase the distinction between number and quantity and, with it, the distinction between real number theory and geometry.

2 Infinity in Greek Mathematics

PREVIEW

Although number and length are mostly kept separate in the *Elements*, there is one process that Euclid applied to both – the Euclidean algorithm, which operates on a pair by 'repeatedly subtracting the smaller from the larger'. When applied to a pair of positive integer numbers (or more generally, to a pair of positive integer multiples of a unit length) the algorithm terminates because positive integers cannot decrease forever. But when applied to a pair of lengths in irrational ratio, the algorithm does not terminate. Indeed, Euclid used non-termination of his algorithm as a criterion for irrationality, thus bringing infinity into the discussion of irrationality.

The number-free theory of area, used by Euclid to prove the Pythagorean theorem, works quite smoothly for areas of polygons. But a similar approach to volume fails for even simple polyhedra, such as the tetrahedron. Euclid was able to find the volume of the tetrahedron by decomposing it into infinitely many prisms, thus bringing infinity into the theory of volume. The Greek theory of area also had difficulty with curved regions, which obviously cannot be decomposed into finitely many polygons. However, Archimedes was able to find the area of a parabolic segment by decomposing it into infinitely many triangles. Nevertheless, the Greeks sought to 'avoid infinity' by considering arbitrary finite sums instead of infinite sums.

2.1 Irrationality and Non-termination

The Euclidean algorithm is introduced in Book VII of the *Elements*, as a method for finding the greatest common divisor of two positive integers. As Euclid says, one 'continually subtracts the less from the greater'; more precisely, if $a > b$, one replaces the pair a,b by $a - b,b$. Since positive integers cannot decrease forever, the algorithm always terminates. For example, with the pair 13, 8, one gets

$$13, 8 \to 5, 8 \to 5, 3 \to 2, 3 \to 2, 1 \to 1, 1.$$

When a pair of identical numbers is obtained, that number is $\gcd(a,b)$, because all common divisors of the pair are preserved by subtraction. Thus our example shows that $\gcd(13, 8) = 1$.

In Book X of the *Elements* Euclid generalizes the algorithm to lengths a and b, in which case it may not terminate. For example, if (using modern notation) the lengths are $a = \sqrt{2}$ and $b = 1$, then the first two steps are

$$\sqrt{2}, 1 \to 1, \sqrt{2} - 1 \to 2 - \sqrt{2}, \sqrt{2} - 1.$$

At this point it may be noticed that $2 - \sqrt{2}$ and $\sqrt{2} - 1$ are in the same ratio as $\sqrt{2}$ and 1. It is not clear whether the Greeks noticed this (though they were probably aware of something similar), but it is clear by basic algebra because

$$2 - \sqrt{2} = \sqrt{2}(\sqrt{2} - 1).$$

Since $2 - \sqrt{2}$ and $\sqrt{2} - 1$ are in the same ratio as $\sqrt{2}$ and 1, applying the Euclidean algorithm to them will produce, in two steps, yet another pair in that ratio – and so on, forever.

Whether or not Euclid knew this particular example, he realized that the algorithm does not terminate on a pair of lengths in irrational ratio (Book X,

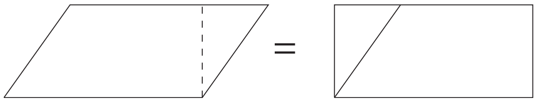

Figure 6 Equality of parallelogram and rectangle.

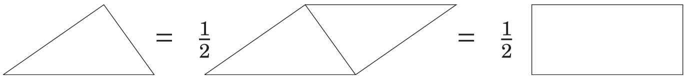

Figure 7 Equality of triangle and half rectangle.

Proposition 2). Thus the Euclidean algorithm elegantly separates rational from irrational, by separating termination from non-termination; that is, finite from infinite.

2.2 Areas and Volumes

In Book I of the *Elements* Euclid shows equality of various regions by adding or subtracting equal triangles. For example, Figure 6 shows that a parallelogram equals a rectangle of the same base and height. And Figure 7 shows that a triangle equals half a rectangle with the same base and height.

There are also decompositions showing that any rectangle equals a rectangle with a given base. Using this fact, it is possible to find a rectangle equal to any polygon, by cutting the polygon into finitely many triangles. Now if (as in the case of the triangle) one region R equals a rational multiple r of some standard region we can take as a unit, then it is compatible with Euclid to let the number r 'measure' the region R in the same way that we measure the area of R. Under these conditions we will speak of numerical 'areas' and 'volumes' from now on.

Now a curved region obviously cannot be cut into finitely many triangles. The best we can hope for is a decomposition into infinitely many triangles which, if we are lucky, might be comprehensible. Archimedes had a brilliant success by this method, finding the area of a parabolic segment.

The parabolic segment is filled with triangles in the manner shown in Figure 8: first the black triangle, then two dark grey triangles below it, then four lighter grey triangles below them, and so on.

Each triangle is half the width of the triangle above it, and a calculation shows that each group (of one, two, four, … triangles) has total area one-fourth the area