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Preface

The *Beyond hyperbolicity* workshop was held at the University of Cambridge Centre for Mathematical Sciences from 20–24 June 2016, with the goal of examining various generalizations of Gromov-hyperbolicity that have recently assumed a prominent role in geometric group theory. The enormous success of the theory of Gromov-hyperbolic spaces and groups in the 30 years since their introduction has inspired geometric group theorists to go "beyond hyperbolicity", i.e., to study groups which are not hyperbolic by exploiting the vestiges of hyperbolicity that they nonetheless exhibit. This paradigm is exemplified by theories that directly generalise hyperbolicity (e.g. relative, acylindrical, and hierarchical hyperbolicity, or the theory of coarse median spaces), and also by ideas reminiscent of the thin-triangle condition that defines a hyperbolic space (e.g. median spaces and various flavours of nonpositive curvature). A major goal of the workshop — and one of our primary aims in creating this volume — was to survey this rich ecosystem of ideas and put them into conversation with one another.

Since a good part of the utility of the theory of hyperbolic groups comes from the notion of the Gromov boundary of a hyperbolic space, it was also our goal to describe the generalisations of the Gromov boundary arising in recent work, notably the contracting boundary, Morse boundary, and hierarchically hyperbolic boundary. Another goal was to look closely at important examples of groups exhibiting various "hyperbolic-like" features: right-angled Artin and other cubulated groups, mapping class groups of surfaces, outer automorphism groups of free groups, et cetera.

The workshop was organised around three mini-courses, whose lecture notes form the basis of the first three articles in this volume: Brian Bowditch (U. Warwick) lectured on coarse median spaces, Martin Bridson (U. Oxford) on semihyperbolicity, and Denis Osin (Vanderbilt U.) on

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acylindrical hyperbolicity. In addition to the mini-courses, there were fifteen lectures encompassing a broad range of recent developments in geometric group theory.

The present volume is based on material addressed in the workshop and aims to provide both a snapshot of the present state of this important branch of geometric group theory and also a reference for those wishing to acquaint themselves with the salient parts of the field. Therefore, in addition to the expository articles based on the mini-courses, we have also included expository articles on two extra topics of current interest: Morse boundaries, and hierarchical hyperbolicity. In addition to the expository articles, there are several research articles representing recent contributions to the theory of groups exhibiting hyperbolic features.

We are very grateful to all of the authors for producing such excellent articles, the anonymous referees for their essential comments, and Benjamin Barrett for meticulously transcribing and beautifully illustrating the mini-course lectures, on which the first three articles in this volume are based. We are also very grateful to those who made the workshop such an exciting and stimulating event: the speakers, the more than 70 participants, the Centre for Mathematical Sciences, and Selwyn College. We finally acknowledge the financial assistance of the Engineering and Physical Sciences Research Council (GRN EP/1003843/2 and EP/L026481/1), which made the workshop possible.

Mark Hagen
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