Introduction

This book deals with a subject that extends synthetic differential geometry [61] to differential topology, in particular to the theory of smooth mappings and their singularities. The setting is that of category theory [86] in general and of topos theory [55] in particular. An excellent introduction to both subjects including applications to several topics (among them synthetic differential geometry) is [79]. The subject of toposes in logic and logic in toposes is illustrated in [21], an article intended for philosophers. Our book is intended as the basis for an advanced course or seminar whose only prerequisite is a reasonable acquaintance with category theory, logic, commutative algebra, infinitesimal calculus, general topology, differential geometry and topology.

Motivated by the desire to employ category theory in a non-trivial way in (elementary) Physics, Lawvere [72] in 1967 gave lectures on 'Categorical Dynamics' which would turn out to be the beginning of a new subject, a branch of (applied) category theory which came to be labelled 'synthetic differential geometry' (SDG), as opposed to 'analytic' which relies heavily on the use of coordinates. What Lawvere proposed was to do Dynamics, not in the context of manifolds, but in a category $\mathscr{E}$, different from the category $\mathscr{M}^\infty$ of smooth paracompact manifolds in several respects : (1) in $\mathscr{E}$, 'the line' would be represented by an object R which, unlike the classical reals, would not be a field but just a commutative ring in which nilpotent elements could be thought of as infinitesimals, and (2) in $\mathscr{E}$, unlike in $\mathscr{M}^\infty$, all finite limits and exponentials would be assumed to exist so that, for the objects of $\mathscr{E}$ thought of as 'smooth spaces' and for the morphisms of $\mathscr{E}$ thought of as 'smooth maps', one could form all fibred products (not just the transversal ones) and something so basic as the smooth space of all smooth maps between two smooth spaces would exist.

The idea of introducing infinitesimals so as to render more intuitive the foundations of analysis was not new. On the one hand, there are non-standard models of analysis [102] in which the non-standard reals have infinitesimals, but where the field property is retained and so the possibility of dividing by non-zero elements gives infinitely large non-zero reals. On the other hand, commutative algebra deals with nilpotent elements in rings and treats them as infinitesimals of some kind. However, on account of the assumptions made about the line, SDG is quite different from non-standard analysis and goes beyond commutative algebra as it has models arising also from differential geometry and analysis and not just from commutative algebra and algebraic geometry.

It is customary to assume further that $\mathscr{E}$ is a topos, even a Grothendieck topos [4], although the Grothendieck toposes that are usually considered as models of SDG are C^∞ versions of those devised by Grothendieck to do algebraic geometry. The idea of working in a topos is not new either as Chen, in 1977, constructed a 'gros' topos for the same purpose, but one in which there was no room for infinitesimals [29]. The two conditions imposed on $\mathscr{E}$ by Lawvere were put to work in SDG by means of the basic axiom of the theory, namely, the axiom that states that R be 'of line type', also known as the 'Kock-Lawvere axiom', and which we discuss in the first part of this book. As stated already, these developments owe much to the lead of André Weil [111] and Charles Ehresmann [40], although the SDG treatment of classical differential geometry differs from those in that the basic constructions in SDG are more natural than in theirs; for instance, tangent spaces are representable as some sort of function spaces, whereas this is not the case in the approach by means of 'near points'.

Although the origins of SDG were strongly influenced by several developments, such as Robinson's non-standard analysis, Weil and Ehresmann's theory of infinitely near points, Grothendieck's use of toposes in algebraic geometry, and Chen's gros toposes in his treatment of the calculus of variations, it differs from all four of them. It differs from non-standard analysis in that SDG is carried out in a topos whose internal logic is necessarily non-Boolean and where R is not a field. It differs from the Weil and Ehresmann's treatment in that the tangent spaces and other spaces of jets are presented as function spaces which need no special construction as they exist naturally by virtue of the topos axioms. It differs from Chen's gros topos models in that in SDG infinitesimals exist and so permit intuitive and direct arguments in the style of non-standard analysis. It differs from the use of Grothendieck toposes in algebraic geometry in that the well adapted models for SDG, by which it is meant models with $\mathscr{E}$ a topos and R a ring of line type in the generalized sense, for which a full embedding $\mathscr{M}^\infty \hookrightarrow \mathscr{E}$ of the category of smooth manifolds exists and has some good properties, such as sending $\mathbb{R}$ to R, preserving limits that exist and constructions that are available, are quite different although in a sense analogous to those arising from the affine schemes in that the smooth aspect and corresponding notion of C^∞-ring is the basis for constructing such models [32, 34, 33].

The introduction of the intrinsic (or Penon) topology [96, 94] on any object of a topos $\mathscr{E}$ and, for a model $(\mathscr{E}, R)$ of SDG, that of the object $\Delta(n) = \neg\neg\{0\} \hookrightarrow R^n$ of 'all infinitesimals' in R^n, intended to represent germs at $0 \in R^n$ of smooth mappings from R^n to R, opened up the way to synthetic differential topology (SDT). In particular, a synthetic theory of stable mappings to be based on SDT was proposed as a theory which extended SDG by means of axioms and postulates (germs representability, tinyness of the representing objects, infinitesimal inversion, infinitesimal integration of vector fields, density of regular values) introduced formally in [20, 24]. The main application of Mather's theorem (infinitesimally stable germs are stable) is a useful tool for the classification of stable mappings. We give two proofs of it here, one which (as in the classical case) makes use of a 'Weierstrass preparation theorem' [26], and another [103] which does not. As in the classical case, the notion of a generic property was introduced in SDT [44] and was shown therein to be satisfied by the stable germs. In the case of Morse germs [46] in SDT, genericity is shown to follow from the facts that Morse germs are both stable and dense.

A general way to proceed in applying SDG or SDT to classical differential geometry or topology is as follows. First, a classical problem or statement is formulated in the internal language of the topos $\mathscr{E}$, where $(\mathscr{E}\ R)$ is a well adapted model of the synthetic theory **T** to be used (for instance SDT or just SDG), in such a way that when applying the global sections functor $\Gamma = \mathrm{Hom}(1, -) \colon \mathscr{E} \longrightarrow \mathit{Set}$ the original problem or statement be recovered. The second step consists in making use of the rich structure of the topos $\mathscr{E}$ (finite limits, exponentiation, infinitesimals) in order to give definitions or prove theorems in a conceptually simpler and more intuitive fashion than in their classical forms. It is often the case that this step requires an enrichment of the synthetic theory **T** through the adoption of additional axioms. A guideline for the selection of such axioms is restricted by the need to ultimately prove their consistency with the axioms of **T**. This requirement renders the subject less trivial than what it may appear at first, as the axioms should also be as few and as basic as possible. The verification of the validity of the additional axioms in $\mathscr{E}$ constitutes the third step. The fourth and final step is to reinterpret the internal solution to the problem as a classical statement, either by applying the global sections functor Γ (which, however, has poor preservation properties in general) or by restricting the objects involved to those that arise from a classical set-up via the embedding $\iota \colon \mathscr{M}^\infty \hookrightarrow \mathscr{E}$.

The applications of SDG to classical differential geometry and topology that are given in this book are to the theory of connections and sprays [28], the calculus of variations [27], the stability theory of smooth mappings [26, 44, 103], and Morse theory [45]. In order to carry out such applications, an acquaintance with the appropriate portions of the subject matter itself is naturally a prerequisite. There are several references where the classical theories of connections and the calculus of variations are expounded. For the former, our sources were [1], [93], and [98]. For the latter, we used [9] and [49]. Among the classical sources for the theory of smooth manifolds and their singularities including Morse theory, our sources were [3], [11], [13], [47], [48], [51], [54], [68], [82], [83, 84, 85], [87], [91], [104], [106], [108], and [110]. In this case, what is needed in order to derive the classical theorems (and generalizations of them) from their versions within SDT is to establish the existence of a well adapted model of the latter. This is precisely what concerns the last part of the book.

This book consists of six parts. In the *first part* we review all basic notions of the theory of toposes that are needed in the sequel. Of particular importance are two such notions that arose in connection with applications of toposes in set theory, algebraic geometry and differential geometry and topology, to wit, atoms and Penon opens. If desired, this material could be extended to cover some of the topics from [55, 6, 12] and references therein. This is followed by a summary of the main aspects of synthetic differential geometry, which we refer to as SDG [61]. The first axiom of SDG postulates, for a topos $\mathscr{E}$ with a natural numbers object N and a commutative ring R in it, the representability of jets of mappings as mappings themselves. As a second axiom we postulate that the jet representing objects be in some sense infinitesimal. To these two axioms we add several postulates that are used in order to develop part of the differential calculus. In order to illustrate the uses of SDG for differential geometry and analysis we give, in the *second part* of this book, two different applications of it: a theory of connections and sprays [28, 22], and a version of the calculus of variations [52, 27]. In the theory of connections and sprays within SDG it is emphasized that, unlike the classical theory, the passage from connections to (geodesic) sprays need not involve integration except in infinitesimal form. In the case of the calculus of variations within SDG, it is shown that one can develop it without variations except for those in an infinitesimal guise. In both illustrations, the domain of application is the class of infinitesimally linear objects, which includes R and is closed under finite limits, exponentiation and étale descent. In particular, and in both cases, the domain of applications of SDG extends beyond their classical counterparts.

In the *third part* of this book we introduce the subject matter of the title. The origin of synthetic differential topology, which we refer to as SDT, can be traced back to the introduction [96] of an intrinsic topological structure on any object of a topos ('Penon opens'). This in turn motivated the introduction and study of general topological structures in toposes [25] and is included here as a preliminary to the specific topological structures of interest in this book, that is, the Euclidean and the weak topological structures. By synthetic differential topology (SDT) we shall understand an extension of synthetic differential geometry (SDG) obtained by adding to it axioms of a local nature—to wit, germ representability and the tinyness of the representing objects [96, 25], which are logical, rather than algebraic infinitesimals. To those, we add four postulates.

The problem of classifying all germs of smooth mappings according to their singularities is intractable. Topologists reduce the question to the consideration of stable (germs of) smooth mappings. In the context of synthetic differential topology, the entire subject is considerably simplified by the force of the axiom of the representability of germs of smooth mappings by means of logical infinitesimals. A smooth mapping is said to be stable if any infinitesimal deformation of it is equivalent to it, in the sense that under a small deformation there is no change in the nature of the function. A class of mappings is said to be generic if the class is closed under equivalence and is dense in that of all smooth mappings equipped with the Whitney topology. The main tool in the classification problem is Mather's theorem [84]. A theory of germs of smooth mappings within SDT has been developed by the authors of the present monograph [20, 44, 26, 103] and constitutes the *fourth part* of this book. The notion of stability for mappings, or for germs, is important for several reasons, one of which is its intended application in the natural sciences, as promoted by R. Thom [107]. Another reason for concentrating on stability is the simplification that it brings about to the classification of singularities. In this same part we apply the results obtained in order to give a version [45, 46] of Morse theory [87] within SDT.

In the *fifth part* of the book we introduce a notion of well adapted model of SDT, based on a previous notion of that of a well adapted model of SDG. In the *sixth part* of the book we focus on a particular model of SDT that is shown to be well adapted to the applications to classical mathematics in the sense of [32, 10]. This model is the Dubuc topos $\mathscr{G}$ [34], constructed using the notion of a C^∞-ring which is due to F.W. Lawvere and goes back to [70]. What makes this topos a well adapted model of SDT (in fact, the only one that is known, at present) is the nature of the ideals, which are germ determined or local. Some of the axioms involved in the synthetic theory for differential topology are intrinsically related to this particular model, whereas others were suggested by their potential applications to a theory of smooth mappings and their singularities. The existence of a well adapted model of SDT is what renders it relevant to classical mathematics.