

Complex Analysis

Second edition

This new edition of a classic textbook develops complex analysis from the established theory of real analysis by emphasising the differences that arise as a result of the richer geometry of the complex plane. Key features of the authors' approach are to use simple topological ideas to translate visual intuition into rigorous proof, and, in this edition, to address the conceptual conflicts between pure and applied approaches head-on.

Beyond the material of the clarified and corrected original edition, there are three new chapters: Chapter 15 on infinitesimals in real and complex analysis; Chapter 16 on homology versions of Cauchy's Theorem and Cauchy's Residue Theorem, linking back to geometric intuition; and Chapter 17 outlines some more advanced directions in which complex analysis has developed, and continues to evolve into the future.

With numerous worked examples and exercises, clear and direct proofs, and a view to the future of the subject, this is an invaluable companion for any modern complex analysis course.

Ian Stewart, FRS is Emeritus Professor of Mathematics at the University of Warwick. He is author or coauthor of over 190 research papers and is the bestselling author of over 120 books, from research monographs and textbooks to popular science and science fiction. His awards include the Royal Society's Faraday Medal, the IMA Gold Medal, the AAAS Public Understanding of Science Award, the LMS/IMA Zeeman Medal, the Lewis Thomas Prize, and the Euler Book Prize. He is an honorary wizard of the Discworld's Unseen University.

David Tall is Emeritus Professor of Mathematical Thinking at the University of Warwick and is known internationally for his contributions to mathematics education. He is author or coauthor of over 200 papers and 40 books and educational computer software, covering all levels from early childhood to research mathematics.

Cambridge University Press
978-1-108-43679-3 — Complex Analysis
Ian Stewart, David Tall
Frontmatter
[More Information](#)

Complex Analysis

(The Hitch Hiker's Guide to the Plane)

Second edition

IAN STEWART

DAVID TALL

University of Warwick



CAMBRIDGE
UNIVERSITY PRESS

Cambridge University Press
978-1-108-43679-3 — Complex Analysis
Ian Stewart, David Tall
Frontmatter
[More Information](#)

CAMBRIDGE UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom
One Liberty Plaza, 20th Floor, New York, NY 10006, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India
79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108436793
DOI: 10.1017/9781108505468

First edition © Cambridge University Press 1983
Second edition © JOAT Enterprises and David Tall 2018

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 1983
Thirteenth reprint 2004
Second edition 2018

Printed in the United Kingdom by TJ International Ltd, Padstow, Cornwall 2018

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data

Names: Stewart, Ian, 1945– author. | Tall, David Orme, author.
Title: Complex analysis : the hitch hiker's guide to the plane / Ian Stewart (University of Warwick), David Tall (University of Warwick).
Other titles: Hitch hiker's guide to the plane
Description: Second edition. | Cambridge : Cambridge University Press, 2018.
| Includes bibliographical references and index.
Identifiers: LCCN 2018007009 | ISBN 9781108436793 (alk. paper)
Subjects: LCSH: Functions of complex variables. | Numbers, Complex. | Geometry, Analytic–Plane.
Classification: LCC QA331 .S85 2018 | DDC 515/.9–dc23
LC record available at <https://lcn.loc.gov/2018007009>

ISBN 978-1-108-43679-3 Paperback

Additional resources for this publication at www.cambridge.org/Stewart&Tall2ed

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

Contents

	<i>Preface to the Second Edition</i>	<i>page xi</i>
	<i>Preface to the First Edition</i>	xiv
0	The Origins of Complex Analysis, and Its Challenge to Intuition	1
	0.1 The Origins of Complex Numbers	1
	0.2 The Origins of Complex Analysis	5
	0.3 The Puzzle	6
	0.4 Is Mathematics Discovered or Invented?	7
	0.5 Overview of the Book	10
1	Algebra of the Complex Plane	13
	1.1 Construction of the Complex Numbers	13
	1.2 The $x + iy$ Notation	15
	1.3 A Geometric Interpretation	16
	1.4 Real and Imaginary Parts	17
	1.5 The Modulus	17
	1.6 The Complex Conjugate	18
	1.7 Polar Coordinates	19
	1.8 The Complex Numbers Cannot be Ordered	20
	1.9 Exercises	21
2	Topology of the Complex Plane	24
	2.1 Open and Closed Sets	26
	2.2 Limits of Functions	27
	2.3 Continuity	30
	2.4 Paths	35
	2.4.1 Standard Paths	35
	2.4.2 Visualising Paths	37
	2.4.3 The Image of a Path	37
	2.5 Change of Parameter	38
	2.5.1 Preserving Direction	39
	2.6 Subpaths and Sums of Paths	39
	2.7 The Paving Lemma	43

vi	Contents	
	2.8 Connectedness	46
	2.9 Space-filling Curves	52
	2.10 Exercises	55
3	Power Series	59
	3.1 Sequences	59
	3.2 Series	63
	3.3 Power Series	66
	3.4 Manipulating Power Series	69
	3.5 Products of Series	71
	3.6 Exercises	72
4	Differentiation	75
	4.1 Basic Results	75
	4.2 The Cauchy–Riemann Equations	78
	4.3 Connected Sets and Differentiability	82
	4.4 Hybrid Functions	83
	4.5 Power Series	84
	4.6 A Glimpse Into the Future	87
	4.6.1 Real Functions Differentiable Only Finitely Many Times	87
	4.6.2 Bad Behaviour of Real Taylor Series	88
	4.6.3 The Blancmange function	89
	4.6.4 Complex Analysis is Better Behaved	91
	4.7 Exercises	92
5	The Exponential Function	96
	5.1 The Exponential Function	96
	5.2 Real Exponentials and Logarithms	98
	5.3 Trigonometric Functions	99
	5.4 An Analytic Definition of π	100
	5.5 The Behaviour of Real Trigonometric Functions	101
	5.6 Dynamic Explanation of Euler’s Formula	103
	5.7 Complex Exponential and Trigonometric Functions are Periodic	104
	5.8 Other Trigonometric Functions	105
	5.9 Hyperbolic Functions	106
	5.10 Exercises	107
6	Integration	111
	6.1 The Real Case	112
	6.2 Complex Integration Along a Smooth Path	113
	6.3 The Length of a Path	117
	6.3.1 Integral Formula for the Length of Smooth Paths and Contours	119
	6.4 If You Took the Short Cut . . .	122
	6.5 Further Properties of Lengths	122

6.5.1	Lengths of More General Paths	123
6.6	Regular Paths and Curves	124
6.6.1	Parametrisation by Arc Length	126
6.7	Regular and Singular Points	127
6.8	Contour Integration	130
6.8.1	Definition of Contour Integral	131
6.9	The Fundamental Theorem of Contour Integration	133
6.10	An Integral that Depends on the Path	136
6.11	The Gamma Function	137
6.11.1	Known Properties of the Gamma Function	139
6.12	The Estimation Lemma	140
6.13	Consequences of the Fundamental Theorem	143
6.14	Exercises	146
7	Angles, Logarithms, and the Winding Number	149
7.1	Radian Measure of Angles	150
7.2	The Argument of a Complex Number	151
7.3	The Complex Logarithm	153
7.4	The Winding Number	155
7.5	The Winding Number as an Integral	159
7.6	The Winding Number Round an Arbitrary Point	159
7.7	Components of the Complement of a Path	160
7.8	Computing the Winding Number by Eye	161
7.9	Exercises	164
8	Cauchy's Theorem	169
8.1	The Cauchy Theorem for a Triangle	171
8.2	Existence of an Antiderivative in a Star Domain	173
8.3	An Example – the Logarithm	175
8.4	Local Existence of an Antiderivative	176
8.5	Cauchy's Theorem	177
8.6	Applications of Cauchy's Theorem	180
8.6.1	Cuts and Jordan Contours	181
8.7	Simply Connected Domains	183
8.8	Exercises	184
9	Homotopy Versions of Cauchy's Theorem	187
9.1	Informal Description of Homotopy	187
9.2	Integration Along Arbitrary Paths	189
9.3	The Cauchy Theorem for a Boundary	191
9.4	Formal Definition of Homotopy	195
9.5	Fixed End Point Homotopy	197
9.6	Closed Path Homotopy	198
9.7	Converse to Cauchy's Theorem	201

9.8	The Cauchy Theorems Compared	202
9.9	Exercises	204
10	Taylor Series	207
10.1	Cauchy Integral Formula	208
10.2	Taylor Series	209
10.3	Morera's Theorem	212
10.4	Cauchy's Estimate	213
10.5	Zeros	214
10.6	Extension Functions	217
10.7	Local Maxima and Minima	219
10.8	The Maximum Modulus Theorem	220
10.9	Exercises	221
11	Laurent Series	225
11.1	Series Involving Negative Powers	225
11.2	Isolated Singularities	230
11.3	Behaviour Near an Isolated Singularity	232
11.4	The Extended Complex Plane, or Riemann Sphere	234
11.5	Behaviour of a Differentiable Function at Infinity	236
11.6	Meromorphic Functions	237
11.7	Exercises	239
12	Residues	243
12.1	Cauchy's Residue Theorem	243
12.2	Calculating Residues	246
12.3	Evaluation of Definite Integrals	248
12.4	Summation of Series	258
12.5	Counting Zeros	261
12.6	Exercises	263
13	Conformal Transformations	268
13.1	Measurement of Angles	268
13.1.1	Real Numbers Modulo 2π	268
13.1.2	Geometry of $\mathbb{R}/2\pi$	269
13.1.3	Operations on Angles	270
13.1.4	The Argument Modulo 2π	270
13.2	Conformal Transformations	271
13.3	Critical Points	276
13.4	Möbius Maps	278
13.4.1	Möbius Maps Preserve Circles	278
13.4.2	Classification of Möbius Maps	279
13.4.3	Extension of Möbius Maps to the Riemann Sphere	281
13.5	Potential Theory	281

	13.5.1	Laplace's Equation	281
	13.5.2	Design of Aerofoils	283
	13.6	Exercises	284
14		Analytic Continuation	289
	14.1	The Limitations of Power Series	289
	14.2	Comparing Power Series	291
	14.3	Analytic Continuation	293
	14.3.1	Direct Analytic Continuation	293
	14.3.2	Indirect Analytic Continuation	295
	14.3.3	Complete Analytic Functions	296
	14.4	Multiform Functions	296
	14.4.1	The Logarithm as a Multiform Function	297
	14.4.2	Singularities	298
	14.5	Riemann Surfaces	299
	14.5.1	Riemann Surface for the Logarithm	299
	14.5.2	Riemann Surface for the Square Root	300
	14.5.3	Constructing a General Riemann Surface by Gluing	301
	14.6	Complex Powers	302
	14.7	Conformal Maps Using Multiform Functions	304
	14.8	Contour Integration of Multiform Functions	305
	14.9	Exercises	311
15		Infinitesimals in Real and Complex Analysis	315
	15.1	Infinitesimals	316
	15.2	The Relationship Between Real and Complex Analysis	318
	15.2.1	Critical Points	320
	15.3	Interpreting Power Series Tending to Zero as Infinitesimals	322
	15.4	Real Infinitesimals as Variable Points on a Number Line	323
	15.5	Infinitesimals as Elements of an Ordered Field	324
	15.6	Structure Theorem for any Ordered Extension Field of \mathbb{R}	327
	15.7	Visualising Infinitesimals as Points on a Number Line	328
	15.8	Complex Infinitesimals	331
	15.9	Non-standard Analysis and Hyperreals	333
	15.10	Outline of the Construction of Hyperreal Numbers	336
	15.11	Hypercomplex Numbers	337
	15.12	The Evolution of Meaning in Real and Complex Analysis	341
	15.12.1	A Brief History	341
	15.12.2	Non-standard Analysis in Mathematics Education	342
	15.12.3	Human Visual Senses	344
	15.12.4	Computer Graphics	345
	15.12.5	Summary	346
	15.13	Exercises	346

16	Homology Version of Cauchy's Theorem	350
	16.0.1 Outline of Chapter	351
	16.0.2 Group-theoretic Interpretation	353
16.1	Chains	354
16.2	Cycles	356
	16.2.1 Sums and Formal Sums of Paths	357
16.3	Boundaries	358
16.4	Homology	360
16.5	Proof of Cauchy's Theorem, Homology Version	362
	16.5.1 Grid of Rectangles	363
	16.5.2 Proof of Theorem 16.2	365
	16.5.3 Rerouting Segments	366
	16.5.4 Resumption of Proof of Theorem 16.2	368
16.6	Cauchy's Residue Theorem, Homology Version	368
16.7	Exercises	370
17	The Road Goes Ever On . . .	374
17.1	The Riemann Hypothesis	374
17.2	Modular Functions	377
17.3	Several Complex Variables	378
17.4	Complex Manifolds	379
17.5	Complex Dynamics	379
17.6	Epilogue	381
	<i>References</i>	382
	<i>Index</i>	383

Preface to the Second Edition

The first edition of *Complex Analysis* focused on generalising concepts from real analysis to the complex case. Where there were differences, we looked at the geometric picture to see why they were happening. This second edition does the same, but it also focuses on the increasing sophistication of mathematical ideas as we build from intuition to rigour, in a manner where greater understanding leads to more sophisticated intuitions and ways of working. New concepts and methods often start out in a technical way, with problematic aspects that conflict with intuition. As well as generalising real analysis, we move beyond it by addressing these conceptual conflicts, resolving them, and providing more sophisticated concepts and methods appropriate to complex analytic functions.

This approach is used throughout the book. So, for example, the text now includes a short (but complete) discussion of the construction of a space-filling curve, to challenge our intuition about continuity and to explain why we have had to be careful with topological assertions that appear obvious. The treatment here is simpler than most of the literature on space-filling curves. We have spent some time examining different notions of a path, especially the role of smoothness.

We have added three new chapters. Chapter 15 introduces ideas about infinitesimals in real and complex analysis, thought of as variables that tend to zero, and formulated as elements of extensions of the real and complex fields. Chapter 16 gives a formal link from analysis back to geometric intuition, formulating and proving homology versions of Cauchy's Theorem and Cauchy's Residue Theorem. Chapter 17 outlines a few of the more advanced directions in which complex analysis has developed, and continues to evolve into the future.

Chapter 15 has been added for the following reasons. Since the first edition appeared in 1983, the ways in which we operate mathematically have changed dramatically. Not only are there computers that perform numerical and symbolic operations at a speed way beyond that previously available to the individual mind; there are also interactive graphics drawn on high-resolution screens that let us visualise mathematical ideas in completely new ways. In particular, we can dynamically magnify pictures to see tiny detail that lets us represent 'arbitrarily small' quantities.

This second edition therefore includes an extra chapter to introduce formally defined infinitesimals that lie in an ordered extension field K of the real numbers, which can be manipulated algebraically and visualised formally on an extended number line. This approach generalises to the complex case using the field $K(i)$ where $i^2 = -1$, which

can be visualised in the extended complex plane. This construction offers a meaningful bridge between the epsilon-delta rigour of pure mathematics and the intuitive use of infinitesimals in applications.

It can easily be shown that any proper ordered field extension K of the reals must contain infinitesimal elements x : that is, elements that are not zero yet satisfy $|x| < r$ for all positive real numbers r . Using the completeness of the real numbers, we prove a simple theorem that any finite element of K has the form $k = c + h$, where c is real and h is infinitesimal or zero. A transformation in the form $m(x) = (x - c)/\varepsilon$, where ε is a positive infinitesimal, then lets us magnify infinitesimal detail near c and see it with our unaided human eyes in a real picture. This technique extends to the complex case in the field $K(i)$.

We can now illustrate why complex analysis is so different from real analysis. A differentiable complex function defined on an open set is locally expressible as a power series, and we may take K to be the smallest ordered extension field generated by a single infinitesimal ε . The elements are power series $\sum_{r \geq n} a^r \varepsilon^r$ in ε with possibly a finite number of terms in $1/\varepsilon$, and each non-zero element has an order of infinitesimality n related to the first non-zero coefficient a_n (where the element may be infinite if n is negative). Meanwhile a differentiable real function may be differentiable once but not twice, and this requires a much more sophisticated extension field K such as that given by the logical theory of non-standard analysis. While Gottfried Leibniz imagined infinitesimals of different orders, non-standard analysis fails to have this property and requires a much more sophisticated construction. At the end of the chapter we compare and contrast the various theories within a single framework.

Chapter 16 on homology complements Chapter 9 on homotopy versions of Cauchy's Theorem, and logically it could have been placed immediately after that. We postpone it to the penultimate chapter because we do not wish to delay the more practical payoff from Cauchy's Theorem – Taylor and Laurent series, residues, evaluation of integrals, summation of series, and so on.

Homology can be thought of as a way of characterising 'holes' in a topological space, which here is the domain of a complex function f . Singularities, where f is not differentiable, create such holes, and homology helps to describe the topological effect of singularities; for example, in the homology version of Cauchy's Residue Theorem. To avoid including big chunks of algebraic topology, our approach to homology is based on step paths in open subsets of the plane, one of the main simplifying tools in this book. The proof is 'bare hands' and exploits the simple geometry of step paths and the abelian group structure of homology.

Chapter 17 has been included to make it clear that complex analysis is still a major area of mathematical research. Complete though the classical theory may seem to be, there are numerous generalisations and new questions. The main topics mentioned are the Riemann Hypothesis, modular functions, several complex variables, complex manifolds, and complex dynamics – leading to the fractal geometry of Julia sets and the famous Mandelbrot set.

In this new edition of *Complex Analysis* we have corrected all known typographical errors, simplified some proofs, and reorganised the material in mostly harmless ways to

improve readability. We have brought the text and layout into line with current practice, and redrawn all the figures. Proofs, definitions, and examples are terminated with the ‘end of proof’ symbol \square . The same symbol indicates the absence of a proof when the result is clear or has already been proved. Contrary to the prevailing wisdom, we do not insert punctuation marks at the end of displayed formulas. (Your tutors may object to this. Tradition is on their side. If they do, they can set you an extra exercise: *insert all missing punctuation*.) But it is now the twenty-first century. No one puts full stops (US: periods) at the end of book titles, or chapter or section headings. So why do this in displayed formulas, where it may cause confusion because punctuation marks are also often part of the symbolism? We suggest that clean typography should override pedantic punctuation.

Formulas in the main text are another matter; here the *absence* of punctuation can cause confusion. We have followed tradition here.

Online Supplementary Material

Supplementary material including a concordance showing in more detail the changes between the previous edition and this one, and links to *GeoGebra*, can be found on the Cambridge University Press website: www.cambridge.org/Stewart&Tall2ed.

Preface to the First Edition

Students faced with a course on ‘Complex Analysis’ often find it to be just that – complex. In the sense of ‘complicated’.

It’s true, of course, that the proofs of some of the major theorems in the subject can demand a certain technical versatility. But in many ways, on a conceptual level, complex analysis is actually *easier* than real analysis; it just isn’t always taught that way.

This book is intended for use at the level of second or third year undergraduates, and it is based on experience accumulated from teaching such courses over the past decade. To exhibit the inherent simplicity of complex analysis we have organised the material around two basic principles: (1) generalise from the real case, and (2) when that reveals new phenomena, use the rich geometry of the plane to understand them. Our aim throughout is to encourage geometric thinking, with the proviso that it must be adequately backed by analytic rigour.

The opening chapter sets the work in its historical context, and the history is often alluded to later as partial motivation. However, we feel that cultural changes often affect the status of conceptual problems: what was once an important difficulty can become a triviality when viewed with hindsight. It is not always necessary to drag today’s students through yesterday’s hang-ups. We argue the point at greater length below: it is fundamental to our entire approach.