TOPOLOGICAL AND NON-TOPOLOGICAL SOLITONS IN SCALAR FIELD THEORIES

Solitons emerge in various nonlinear systems – from nonlinear optics and condensed matter to nuclear physics, cosmology, and supersymmetric theories – as stable, localized configurations behaving in many ways like particles. This book provides an introduction to integrable and non-integrable scalar field models with topological and non-topological soliton solutions. It brings together discussion of solitary waves and construction of soliton solutions in various models and provides a discussion of solitons using simple model examples, including the Kortenweg–de Vries system, the sine-Gordon model, kinks, oscillons, skyrmions, and hopfions. The classical field theory of the scalar field in various spatial dimensions is used throughout to present related concepts, at technical and conceptual levels. Providing a comprehensive introduction to the description and construction of solitons, this book is ideal for researchers and graduate students in mathematics and theoretical physics.

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Preface

There have been many remarkable developments in modern mathematics and theoretical physics over the past 60 years. In particular, one of the greatest breakthroughs occurred in our understanding of nonlinear phenomena. In fact, until the 1960s, nonlinear systems were barely considered; even now, a graduate student studying physics may easily have the impression that nonlinear systems are somehow anomalous. Indeed, the equations of Newtonian mechanics are linear, as are Maxwell's equations, quantum mechanics, and quantum electrodynamics. Our world is full of nonlinear phenomena, so the reality is that linear models are usually too simple to describe a wide variety of physical situations.

One reason nonlinear systems were given little attention is related to their complexity, as most of the corresponding dynamical equations do not possess analytical solutions. Even the superposition principle, which is well known from physics textbooks, cannot be applied in nonlinear theories. The situation changed drastically with the dawning of computational physics, which made it possible to find reasonably accurate solutions for nearly any properly formulated physical problem.

Substantial progress has been made in the last 50 years in understanding properties of various nonlinear systems that arise in many different areas of physics, e.g., physics of plasma, solid-state physics, nonlinear optics, biophysics, and field theory. Further, these developments have sparked a lot of interest in the mathematical investigation of nonlinear systems, with fascinating techniques and applications becoming important mathematical tools. Investigation of these systems reveals many interesting mathematical structures, which surprisingly also appear in quantum field theory and condensed matter physics. From a pragmatic point of view, these nonlinear models provide a substantial extension of a physicist's "tool kit" that otherwise is mainly restricted to solving linear systems.

At first glance, one could naively take into account various nonlinearities, treating them as perturbations in a linear system. This perturbative approach, however, is completely misleading; it does not capture the most important features of nonlinear interactions. In particular, in such a consideration we miss the solitons, which appear as stable, non-dissipative, localized configurations, behaving in many ways like particles. In some situations, their existence is related to topological properties of the model; in other cases they appear due to balance between the effects of nonlinearity and dispersion.

Both topological and non-topological solitons are a good subject for demonstrating a beautiful interplay between mathematics and physics. I believe that an introduction to the basic ideas and techniques related to the description and construction of solitons may be useful to physicists and mathematicians interested in the modern developments in this direction.

Traditionally, there are two main approaches in discussions of soliton configurations in nonlinear systems. The first, which originates from mathematical theory of solitary waves, deals with the concept of integrability and its applications in various models. Another direction is related to the construction of multisoliton solutions of these systems.

The general strategy of this book is to provide an elementary discussion of both approaches on some simple model examples. The classical field theory of real scalar fields in various spatial dimensions seems to be tailor made to present the related concepts. This approach allows us to combine the mathematical discussion of the solitary wave solutions, considering them as non-topological solitons, and the more physical consideration of the topological solitons in field theory. On the other hand, restriction to the scalar models simplifies our discussion significantly.

The book is divided into three parts, according to the spatial dimensionality of the models considered. In Part I I start with a review of the sine-Gordon model. This simple model in one spatial dimension provides an ideal playground to begin our consideration. It supports topological solitons, kinks, and breathers, and is an example of an integrable system.

Kinks in the non-integrable scalar field models with polynomial potentials are discussed in Chapter 2, where I consider the ϕ^4 model. I compare the properties and dynamics of the soliton solutions of this theory, the kinks and the oscillons, with corresponding sine-Gordon solitons. In Chapter 2 I also discuss chaotic dynamics of solitons in non-integrable models.

In Chapter 3, I discuss solitary wave solutions of the Korteweg–de Vries equation, which provide an example of non-topological solitons. To establish a link with the two previous chapters, I consider the Lagrangian and Hamiltonian formulations of the Korteweg–de Vries system and discuss the issue of integrability. In Chapter 3, I also pay some attention to the compacton solutions of the modified Korteweg–de Vries equation, because solutions of that type will be discussed later.

Part II begins with discussion of the $O(3)$ nonlinear sigma model. In Chapter 4, I discuss the restrictions of Derrick's theorem, the idea of topological classification, and reformulation of the nonlinear sigma model in terms of the $\mathbb{C}P^1$ complex variables. Chapter 5 is devoted to the soliton solutions of the planar Skyrme model, so-called baby Skyrmions. I describe how to construct various multisoliton configurations and discuss the pattern of interaction between the solitons in the model with different potentials.

Part III is mostly related to the solitons in three spatial dimensions. First, in Chapter 6, I present stationary Q-balls and discuss their properties. To simplify the discussion I start from the relatively simple case of the one-dimensional Q-ball, which can be written in a closed form. Then I explore the properties of higher-dimensional Q-balls in the original, two-component Friedberg–Lee–Sirlin model, as well as in the complex scalar theory with a polynomial potential.

Chapter 7 contains a survey of soliton configurations in the Skyrme model. I present the rational map construction, describe properties of the Skyrme crystal, and explore the sphaleron solutions of the Skyrme model. I also discuss the procedure of the semiclassical quantization of the spinning multisoliton configurations with restrictions imposed by the generalized Finkelstein–Rubinstein constraints. In Chapter 8, I present the knotted solutions of the Faddeev–Skyrme model and the rational map approximation, and briefly discuss the properties of the related deformed models, like the Nicole model and Aratyn–Ferreira–Zimerman model.

An obvious omission that I do not discuss is solitons in gauged models. My motivation is to provide a compact and coherent introduction to this subject. Consideration of the solitons in the gauge theories, like vortices, monopoles, and instantons, is much more involved. I do not know a better book to read for this purpose than the excellent monograph by N. Manton and P. Sutcliffe, *Topological Solitons* [282], which provides the reader with a solid framework in the modern classical theory of solitons in a very general context. Another omission is the mathematically rigorous inverse scattering problem technique and construction of exact soliton solutions in integrable theories. I have made no attempt to discuss this direction because it would make the presentation much more involved. There are plenty of books where the reader can find a detailed introduction to this area. My intention is to present the minimal body of knowledge on solitons.

Though extensive, the list of references at the end of the book cannot be considered an exhaustive bibliography on solitons. I apologize to those authors whose contributions are not mentioned.

I especially thank Nick Manton, who played a very important role in my understanding of solitons, both through his papers and in private discussions. I take this opportunity to express my deep gratitude to him.

This book is an outgrowth of the lecture notes for a semester graduate course that was originally given at the School of Mathematics, Trinity College Dublin in 2008. I owe a special debt to Sergey Cherkis, who suggested that it might be a good idea to try to combine discussion of both mathematical and field-theoretical aspects of solitons in a single introductory course. I taught similar courses in recent years at Jagiellonian University in Krakow, Poland, the University of Vilnius in Lithuania, the Institute of Physics of São Carlos at the University of São Paulo, Brazil, and Carl von Ossietzky University of Oldenburg, Germany. I am very grateful to many students for critical comments and profound questions.

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