Part I

Kinks and Solitary Waves

1
Sine-Gordon Model

1.1 The Dawning of Solitons: From the Frenkel–Kontorova Model to the Sine-Gordon Model

It is traditional to start a discussion of the solitons with a famous story about John Scott Russell, who first observed and described the solitary waves. Many books on solitons begin from this starting point (see, e.g., [310] or [383]). Here, however, we take another route, departing from the Frenkel–Kontorova model introduced in 1938 [155].[1] This one-dimensional toy model describes a chain of particles, coupled by the horizontal springs to the nearest neighbors and placed in a periodic potential, which, for example, represents a substrate.

As with every good toy model, it has a lot of other realizations. For example, it can be visualized as a system of two parallel superconducting wires with a Josephson junction in between, or even as a model of the basic functions of DNA (see, e.g., [115]).

Here we consider another mechanical analog of the Frenkel–Kontorova model, which was suggested by Scott in 1969 [352]. This is a chain of identical simple pendulums of length l and mass m separated by distance a. The pendulums are oscillating in parallel planes and are elastically coupled through the identical torque springs with their nearest neighbors (see Figure 1.1). Thus, the nth pendulum both vibrates near its equilibrium point x_n, $i = 1, 2, \dots s$, $s \to \infty$ and oscillates under force of gravity.

As a dynamic variable we can consider the deviation angle $\phi(x_n, t)$ from the lower vertical position at time t, then the potential energy of the elastic interaction between two adjacent pendulums is

$$\frac{\alpha}{2}[\phi(x_n) - \phi(x_{n-1})]^2,$$

where α is the torsion constant.

[1] For a detailed review of the model and its applications, see, e.g., [63, 85].

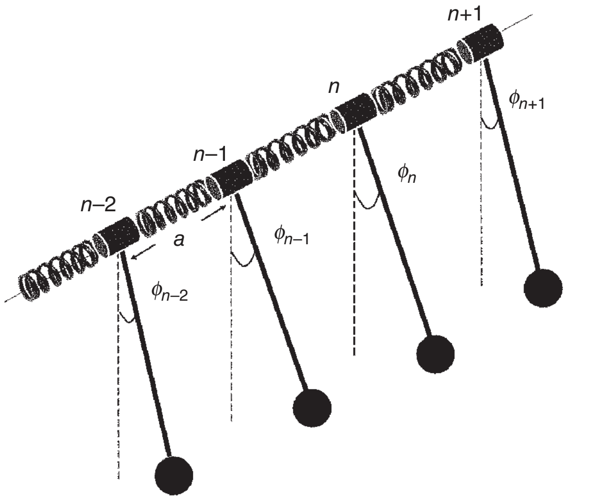

Figure 1.1 Chain of pendulums elastically coupled with their nearest neighbors.

Then the rotational kinetic energy T and the total potential energy U of the system are given by the sum over all the pendulums

$$\begin{aligned} T &= \frac{I}{2}\sum_n \left(\frac{\partial\phi(x_n,t)}{\partial t}\right)^2; \\ U &= \frac{\alpha}{2}\sum_n [\phi(x_{n+1},t)-\phi(x_n,t)]^2 + \sum_n V[x_n,t], \end{aligned} \tag{1.1}$$

where I is moment of inertia of a pendulum and x_n is the coordinate of the nth pendulum in the chain. The external potential $V[x_n]$ is simply the gravitational potential energy

$$V[x_n] = -mgl\,(1-\cos\phi(x_n,t))\,. \tag{1.2}$$

Then the equation of motion of a pendulum placed at x_n is

$$I\frac{\partial^2\phi(x_n,t)}{\partial t^2} - \alpha[\phi(x_{n+1},t)-2\phi(x_n,t)+\phi(x_{n-1},t)] + mgl\sin\phi(x_n,t) = 0. \tag{1.3}$$

Evidently, for small-angle oscillations $\phi(x_n) \ll 1$ this equation can be linearized

$$I\frac{\partial^2\phi(x_n,t)}{\partial t^2} - \alpha[\phi(x_{n+1},t)-2\phi(x_n,t)+\phi(x_{n-1},t)] + mgl\phi(x_n,t) = 0,$$

and the motion is simple harmonic. If we neglect the gravity force, the problem is reduced to

$$I\frac{\partial^2\phi(x_n,t)}{\partial t^2} - \alpha[\phi(x_{n+1},t)-2\phi(x_n,t)+\phi(x_{n-1},t)] = 0. \tag{1.4}$$

This equation can be solved by Fourier transform. Let us multiply (1.4) by e^{-ikn} and sum over all n supposing that $\phi_n \equiv \phi(x_n, t)$ decays to zero for large n:

$$I\frac{\partial^2}{\partial t^2}\sum_{n=-\infty}^{\infty}\phi_n e^{-ikn} = 2\alpha(\cos k - 1)\sum_{n=-\infty}^{\infty}\phi_n e^{-ikn}. \tag{1.5}$$

This is the simple differential equation that allows us to find the coefficients of the Fourier transform, the *kth mode* of the oscillations

$$\phi(k,t) = \sum_{n=-\infty}^{\infty}\phi_n e^{-ikn}.$$

Clearly, the solution is

$$\phi(k,t) = A(k)\cos(\omega t) + B(k)\sin(\omega t),$$

where the frequency $\omega = \sqrt{\frac{2\alpha}{I}(1-\cos k)}$ and $A(k)$, $B(k)$ are arbitrary constants that define the energy of the kth mode. Note that the modes of the linear system are decoupled from each other; there are no transitions between them.

However it is not so easy to find a solution for (1.3) when the linear approximation cannot be used. As a matter of fact one has to apply numerical methods to solve it.

On the other hand, we can consider long-wave excitations in this system. That is, the excitations such that the characteristic length at which ϕ changes significantly is much greater than the distance between neighboring pendulums a. This allows us to introduce the continuum limit of the model (1.1) replacing the discrete variable x_n with the coordinate $x = na$ and then taking the limit $a \to 0$.

The Taylor expansion of the functions $\phi(x_{n+1}) = \phi(x_n + a)$ and $\phi(x_{n-1}) = \phi(x_n - a)$ yields

$$\phi(x_{n\pm1}) \approx \phi(x_n) \pm a\frac{\partial u(x_n)}{\partial x} + \frac{a^2}{2}\frac{\partial^2\phi(x_n)}{\partial x^2} \pm \frac{a^3}{3!}\frac{\partial^3\phi(x_n)}{\partial x^3} + \frac{a^4}{4!}\frac{\partial^4\phi(x_n)}{\partial x^4} + \cdots \tag{1.6}$$

Thus, in the order up to $\mathcal{O}(a^4)$,

$$\phi(x_{n+1}) - 2\phi(x_n) + \phi(x_{n-1}) \approx a^2\frac{\partial^2\phi(x_n)}{\partial x^2},$$

and the equation of motion (1.3) takes the form

$$I\frac{\partial^2\phi(x,t)}{\partial t^2} - \alpha a^2\frac{\partial^2\phi(x,t)}{\partial x^2} + mgl\sin\phi(x,t) = 0. \tag{1.7}$$

We can now introduce the dimensionless variables to absorb the parameters of the model:

$$x \to= \frac{x}{a}\sqrt{\frac{mgl}{\alpha}}; \qquad t \to t\sqrt{\frac{mgl}{I}}.$$

Then (1.7) finally takes the form

$$\frac{\partial^2\phi(x,t)}{\partial t^2} - \frac{\partial^2\phi(x,t)}{\partial x^2} + \sin\phi(x,t) = 0. \tag{1.8}$$

In other words, in the continuum limit $a \to 0$ the set of the discrete real angular variables $\phi(x_n, t)$ becomes the *scalar field* $\phi(x,t)$, which is a continuous canonical variable defined for any coordinate x at any moment of time t.

Equation (1.8), known as the *sine-Gordon equation*, was actually well known long before it got this name. Historically, it was Jacques Edmond Bour [81] who analyzed this equation, considering the compatibility conditions for the Gauss equations for pseudospheres. It was rederived independently by Bonnet in 1867 and Enneper in 1868, again in the context of the differential geometry of surfaces of a constant negative Gaussian curvature.[2]

Consequent study of this equation by Bianchi (1879) and Bäcklund (1880s) [62] resulted in discovery of the interesting result that it is possible to generate a tower of new solutions of (1.8) from one particular known solution, even a trivial one. In Section 1.2 we briefly consider this approach, which is known as the *Bäcklund transformation*.

Furthermore, (1.8) supports solitonic solutions, the *kinks* that we discuss in Section 1.2. To the best of our knowledge, these solutions were first found in 1950 in further consideration of the Frenkel–Kontorova model [248], once again a long time before the idea of solitons became fashionable.

In 1962, Perring and Skyrme [319] formulated the sine-Gordon model as a simple, relativistic, nonlinear scalar field theory. Their description is most appropriate for our discussion.

Equation (1.8) may be derived from the Lagrangian

$$L = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - U[\phi] \equiv \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi), \tag{1.9}$$

where for the sake of generality we introduce the potential of the scalar field $U(\phi) = (1 - \cos\phi)$. Hereafter we make use of the covariant notations in $1+1$ dimensions to make manifest the Lorenz invariance of the model. Our choice for the metric in 1+1 dimensions is $g_{\mu\nu} = \mathrm{diag}\,(1, -1)$ and we adopt the natural units $\hbar = c = 1$ to simplify our notations. The corresponding variational equation is

$$\frac{\partial}{\partial x_\mu}\left(\frac{\delta L}{\delta(\partial_\mu\phi)}\right) = \frac{\delta L}{\delta\phi}, \tag{1.10}$$

which yields the covariant form of (1.8)

$$\partial_\mu\partial^\mu\phi = -U'(\phi). \tag{1.11}$$

[2] There were (unsuccessful) attempts to restore historical credit, e.g., referring to this equation to as the *Enneper (sine-Gordon) equation* [358].

Evidently, the canonical stress energy tensor is

$$T_{\mu\nu} = \left(\frac{\delta L}{\delta(\partial^\mu \phi)}\right)\partial_\nu \phi - g_{\mu\nu}L = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}L. \tag{1.12}$$

Explicitly, the components of $T_{\mu\nu}$ are

$$\begin{aligned} T_{00} &= \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + 1 - \cos\phi; \quad T_{01} = \frac{\partial^2\phi}{\partial x\,\partial t};\\ T_{11} &= \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - 1 + \cos\phi; \quad T_{10} = \frac{\partial^2\phi}{\partial x\,\partial t}. \end{aligned} \tag{1.13}$$

As usual, this tensor is conserved due to translational symmetry of the sine-Gordon model, i.e.,

$$\partial_\mu T^{\mu\nu} = \begin{cases} \partial_t T^{00} - \partial_x T^{10} = 0\\ \partial_t T^{01} - \partial_x T^{11} = 0. \end{cases} \tag{1.14}$$

Since we are interested in finite-energy solutions, we have to consider the total energy of this system

$$E = \int_{-\infty}^{\infty} dx T_{00} = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\partial_x\phi)^2 + U(\phi)\right]. \tag{1.15}$$

The corresponding vacuum solutions of the field equation (1.11) are configurations ϕ_0, which satisfy the stationary points of the action, i.e., we shall search for fields that satisfy the conditions

$$U'(\phi_0) = 0; \qquad U''(\phi_0) > 0. \tag{1.16}$$

The potential of the sine-Gordon model $U(\phi) = (1 - \cos\phi)$ is periodic. It has an infinite number of degenerate vacua at $\phi_0 = 2\pi n, \quad n \in \mathbb{Z}$, for each of those $U''(\phi_0) = 1$.

Lorentz invariance of the model (1.9) allows us to start from the static configurations; they can be boosted if necessary. Therefore, we suppose that $\partial_0\phi = 0$ and the energy functional (1.15) can be written as

$$E = \int_{-\infty}^{\infty} dx \left[\frac{1}{\sqrt{2}}\partial_x\phi \pm \sqrt{U(\phi)}\right]^2 \mp \int_{-\infty}^{\infty} dx\sqrt{2U(\phi)}\ \partial_x\phi \geqslant 0. \tag{1.17}$$

Evidently, the energy is minimal if

$$\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 = U(\phi). \tag{1.18}$$

We suppose that the potential is positively defined for any values of $\phi(x)$, thus we can define a superpotential $W(\phi)$ as a function associated with $U(\phi)$:

$$\frac{1}{2}\left(\frac{\partial W}{\partial \phi}\right)^2 = U(\phi). \tag{1.19}$$

Hence, the second term in (1.17) can be written as

$$\int\limits_{-\infty}^{\infty} dx\sqrt{2U(\phi)}\;\partial_x\phi = W[\phi(\infty)] - W[\phi(-\infty)]. \tag{1.20}$$

Then the lower energy bound (so-called *Bogomolny bound* [77]) is saturated if $E \geqslant W[\phi(\infty)] - W[\phi(-\infty)]$ and the scalar field satisfies the first-order equation

$$\frac{\partial \phi}{\partial x} = \pm\frac{\partial W}{\partial \phi}. \tag{1.21}$$

The idea of superpotential $W(\phi)$ actually originates from supersymmetric models where this function becomes a fundamental quantity (see, e.g., [354]). On the other hand, the method of superpotential is very useful to construct nontrivial soliton solutions in a system of coupled scalar fields [60].

A trivial vacuum solution of the field equation (1.8) is simply the state $\phi = \phi_0 = \text{const}$, where ϕ_0 are the minima of the potential energy $U(\phi)$. Then the Bogomolny bound is saturated trivially, i.e., $W[\phi(\infty)] = W[\phi(-\infty)]$ and $E \geqslant 0$.

We can consider small oscillations about the vacuum, then the Taylor expansion yields $U(\phi - \phi_0) \approx \frac{1}{2}(\phi - \phi_0)^2$ and we arrive at the original linear Klein–Gordon equation for the scalar field of unit mass:

$$(\partial_t^2 - \partial_x^2 + 1)\phi = 0. \tag{1.22}$$

A plane-wave solution of this equation, commonly referred to as a *mode*, is

$$\phi_{k,\omega}(x,t) = Ae^{i(kx-\omega t)}, \tag{1.23}$$

where A is the amplitude, k is the wavenumber of the mode and ω is the frequency of the propagating wave. Substitution of this function into (1.22) yields the dispersion relation

$$-\omega^2 + k^2 + 1 = 0. \tag{1.24}$$

In other words, the linear waves of different lengths propagate with different speeds. Thus, a general solution of the Klein–Gordon equation can be written as an integral sum over all modes

$$\phi(x,t) = \int\limits_{-\infty}^{\infty} dx \left\{A_+(k)e^{i(kx-\omega_+ t)} + A_-(k)e^{i(kx-\omega_- t)}\right\}, \tag{1.25}$$

where $\omega_\pm = \pm\sqrt{k^2+1}$. Evidently, this is a usual expansion in a Fourier series. These states belong to the perturbative sector of the model, in the context of the discrete Frenkel–Kontorova model (1.1) these linear excitations can be identified with phonons.

The solution of the sine-Gordon equation is trivial if the field is in the vacuum state, i.e., $\phi_0 = 2\pi n,\ \ n \in \mathbb{Z}$. Since we are looking for a regular solution with finite total energy, the field must approach the vacuum as $x \pm \infty$ and $\partial_x\phi(x,t) \to 0$ as $x \to \pm\infty$. We also suppose that $\partial_t\phi(x,t)$ is bounded on both ends of the infinite one-dimensional space.

However, the vacuum is infinitely degenerated and the corresponding vacua can be different. For example, we can consider the asymptotic conditions $\phi(-\infty) = 0$ and $\phi(\infty) = 2\pi$. Then the field is not in the vacuum everywhere; it is interpolating between these two vacuum values and the corresponding potential energy of the configuration is no longer zero. Note that in that case the transition to the trivial solution is not possible–the boundary conditions on the field are different for these configurations and it would take an infinite amount of energy to overcome the barrier between these two sectors.[3]

To find nontrivial solutions of the sine-Gordon equation (1.8) let us consider the first-order equation (1.18). Evidently, for the case under consideration $U(\phi) = (1-\cos\phi)$ and the superpotential is $W(\phi) = -4\cos\frac{\phi}{2}$. Thus, the minimal energy bound is saturated if

$$\frac{\partial\phi}{\partial x} = \pm 2\sin\frac{\phi}{2}. \tag{1.26}$$

Separating the variables, we arrive to

$$dx = \pm\frac{d(\phi/2)}{\sin(\phi/2)}. \tag{1.27}$$

Let us consider the positive sign in the right-hand side of this equation. Then the integration yields

$$x - x_0 = \ln\tan\frac{\phi}{4},$$

where x_0 is the integration constant. Thus, we get the nontrivial solution to the sine-Gordon model

$$\phi_K(x) = 4\arctan e^{x-x_0}. \tag{1.28}$$

This solution is referred to as the *kink*. It corresponds to the transition between two neighboring vacua, as at $x \to -\infty$ the field is taking the value $\phi(-\infty) = 0$ while at $x \to \infty$ it approaches another vacuum value, $\phi(\infty) = 2\pi$ (see Figure 1.1).

[3] Strictly speaking, this energy is proportional to the volume of the 1-dim space L.

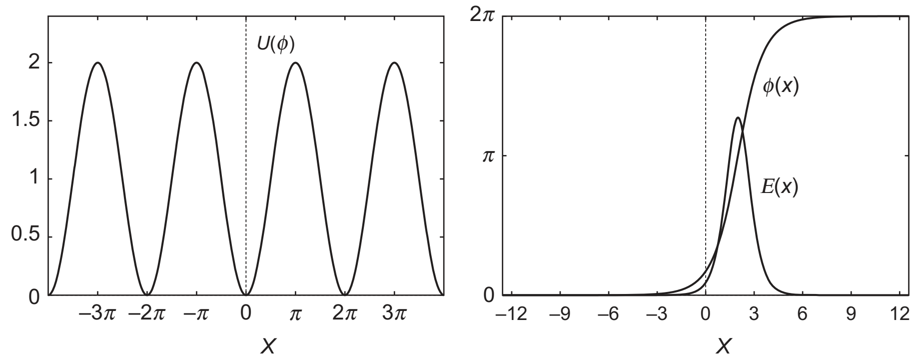

Figure 1.2 The potential of the sine-Gordon model (left panel), the energy density of the static kink, and the kink solution localized at $x_0 = 2$ (right panel).

Obviously, taking the negative sign in the right-hand side of (1.27) we obtain another solution, which interpolates between $\phi(-\infty) = 2\pi$ and $\phi(\infty) = 0$, the antikink

$$\phi_{\bar{K}}(x) = 4\arctan e^{-(x-x_0)}. \tag{1.29}$$

Both kink and antikink are exponentially localized lumps of energy centered around the $x = x_0$. Indeed, substitution of the static solutions (1.28) or (1.29) into the integrand of (1.15) yields the energy-density distribution displayed in Figure 1.2, right panel.

$$E(x) = \frac{4}{\cosh^2(x-x_0)}. \tag{1.30}$$

The dimensionless energy of the static configuration, i.e., its mass, is finite,

$$M = \int_{-\infty}^{\infty} dx E(x) = 8.$$

This solution is an example of a *soliton*, a spacially localized particle-like configuration that is stable and, in many respects, behaves like a particle. The kink state belongs to the non-perturbative sector of the sine-Gordon model; it cannot be obtained via perturbative expansion in the vicinity of a particular vacuum since it becomes infinitely heavy in the weak-coupling limit. Furthermore, the kink solution is a topological soliton. We can introduce the topological current

$$j_\mu = \frac{1}{2\pi}\varepsilon_{\mu\nu}\partial^\nu\phi; \qquad \partial^\mu j_\mu = \frac{1}{2\pi}\varepsilon_{\mu\nu}\partial^\mu\partial^\nu\phi \equiv 0, \tag{1.31}$$