Cambridge University Press & Assessment
978-1-108-42985-6 — Tensor-Valued Random Fields for Continuum Physics
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Introduction

Why Tensor-Valued Random Fields in Continuum Physics?

In this book, we use the term *continuum physics* to refer to continuum mechanics and other classical (non-quantum, non-relativistic) field-theoretic models such as continuum thermomechanics (e.g. thermal conductivity, thermoelasticity, thermodiffusion), electromagnetism and electromagnetic interactions in deformable media (e.g. piezoelectricity). Most tensor-valued (or, in what follows, just 'tensor') fields appearing in these models fall into one of two categories: fields of dependent quantities (displacement, velocity, deformation, rotation, stress...) or fields of constitutive responses (conductivity, stiffness, permeability...). All of these fields take values in linear spaces of tensors of first or higher rank over the space $\mathbb{R}^d$, $d = 2$, 3 and, generally, of random nature (i.e. displaying spatially inhomogeneous, random character), indicating that the well-developed theory of scalar random fields has to be generalised to tensor random fields (TRFs).

In deterministic theories of continuum physics we typically have an equation of the form

$$\mathcal{L}\boldsymbol{u} = \boldsymbol{f},$$

defined on some subset $\mathcal{D}$ of the d-dimensional affine Euclidean space E^d, where $\mathcal{L}$ is a differential operator, $\boldsymbol{f}$ is a *source or forcing function*, and $\boldsymbol{u}$ is a solution field. This needs to be accompanied by appropriate boundary and/or initial conditions. (We use the symbolic ($\boldsymbol{u}$) or, equivalently, the subscript ($u_{i\dots}$) notations for tensors, as the need arises; also, an overdot will mean the derivative with respect to time, $\mathrm{d}/\mathrm{d}t$.)

A field theory is stochastic if either the operator $\mathcal{L}$ is random, or there appears an apparent randomness of $\boldsymbol{u}$ due to an inherent non-linearity of $\mathcal{L}$, or the forcing and/or boundary/initial conditions are random. While various combinations of these basic cases are possible, in this book we focus on the first and second cases.

The first case is typically due to the presence of a spatially random material microstructure; see Ostoja-Starzewski (2008). For example, the coefficients of $\mathcal{L}(\omega)$, such as the elastic moduli $\boldsymbol{C}$, form a tensor-valued random field, and the stochastic equation

$$\mathcal{L}(\omega)\,\boldsymbol{u} = \boldsymbol{f}$$

governs the response of a *random medium* $\mathcal{B}$, that is, the set of possible states of a deterministic medium.

The second case is exemplified by solutions of the Navier–Stokes equation, which becomes so irregular as to be treated in a stochastic way (Batchelor 1951; Monin & Yaglom 2007*a*; Monin & Yaglom 2007*b*; Frisch 1995). In both cases, $\mathcal{B}$ is taken as a set of all the realisations $B(\omega)$ parameterised by elementary events ω of the Ω space

$$\mathcal{B} = \{ B(\omega) : \omega \in \Omega \} . \tag{0.1}$$

In principle, each of the realisations follows deterministic laws of classical mechanics; probability is introduced to deal with the set (0.1). The ensemble picture is termed *stochastic continuum physics*. Formally speaking, we have a triple $(\Omega, \mathfrak{F}, \mathsf{P})$, where Ω is the set of elementary events, $\mathfrak{F}$ is its σ-field and P is the probability measure defined on it.

Besides turbulence, another early field of research where *stochastic continuum physics* replaced the deterministic picture has been *stochastic wave propagation*: elastic, acoustic and electromagnetic. A paradigm of wave propagation in random media is offered by the *wave equation* for a scalar field u in a domain $\mathcal{D}$:

$$\nabla^2 \varphi = \frac{1}{c^2(\omega, \boldsymbol{x})} \frac{\partial^2 \varphi}{\partial t^2}, \qquad \omega \in \Omega, \quad \boldsymbol{x} \in \mathcal{D}.$$

Here c is the wave speed in a linear elastic, isotropic medium, so that, effectively, $\mathcal{B}$ is described by a random field $\{ c(\omega, \boldsymbol{x}) \colon \omega \in \Omega, \boldsymbol{x} \in \mathcal{D} \}$. Given that we simply have a Laplacian on the left-hand side, this model accounts for spatial randomness in mass density ρ only.

In order also to account for randomness in the elastic modulus E, we should consider this partial differential equation:

$$\boldsymbol{\nabla} \cdot [E(\omega, \boldsymbol{x}) \nabla u] = \rho(\omega, \boldsymbol{x}) \frac{\partial^2 u}{\partial t^2}, \qquad \omega \in \Omega, \quad \boldsymbol{x} \in \mathcal{D}. \tag{0.2}$$

Clearly, we are now dealing with two scalar random fields: E and ρ. This model's drawback, however, is the assumption of an inhomogeneous but locally isotropic second-rank stiffness (or elasticity) tensor field $\mathsf{\boldsymbol{E}} = E\mathsf{\boldsymbol{I}}$ instead of $\mathsf{\boldsymbol{E}}$ $(= E_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j)$ with full anisotropy. In fact, extensive studies on upscaling of various mechanical and physical phenomena have shown (Ostoja-Starzewski et al. 2016) that the local anisotropy goes hand in hand with randomness: as the smoothing scale (i.e. scale on which the continuum is set up) increases, the anisotropy and random fluctuations in material properties jointly go to zero. Thus, Equation (0.2) should be replaced by

$$\boldsymbol{\nabla} \cdot [\boldsymbol{E}(\omega, \boldsymbol{x}) \cdot \boldsymbol{\nabla} u] = \rho(\omega, \boldsymbol{x}) \frac{\partial^2 u}{\partial t^2}, \qquad \omega \in \Omega, \quad \boldsymbol{x} \in \mathcal{D}.$$

The same arguments apply to a diffusion equation of, say, heat conduction

$$\boldsymbol{\nabla} \cdot [\boldsymbol{K}(\omega, \boldsymbol{x}) \cdot \boldsymbol{\nabla} T] = c(\omega, \boldsymbol{x}) \rho(\omega, \boldsymbol{x}) \frac{\partial T}{\partial t}, \qquad \omega \in \Omega, \quad \boldsymbol{x} \in \mathcal{D},$$

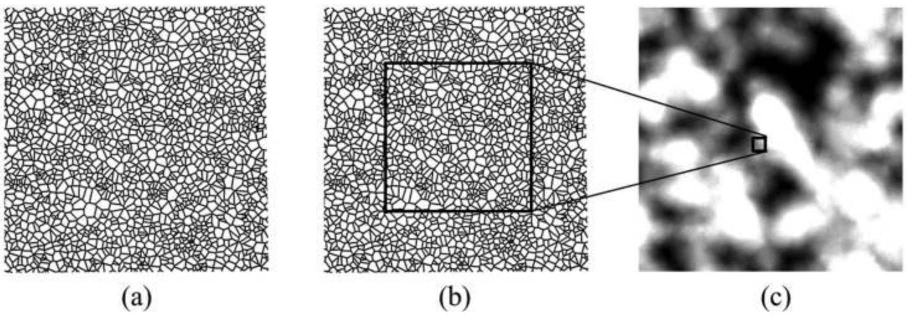

Figure 0.1 (a) A realisation of a Voronoi tesselation (or mosaic); (b) placing a mesoscale window leads, via upscaling, to a mesoscale random continuum approximation in (c). Reproduced from Malyarenko & Ostoja-Starzewski (2017*b*).

in which $\boldsymbol{K}$ is the thermal conductivity tensor (again with anisotropy present), while the specific heat c and mass density ρ jointly premultiply the first derivative of temperature T on the right-hand side.

This line of reasoning also applies to elliptic problems: consider Figure 0.1, showing a planar Voronoi tessellation of E^2 which serves as a planar geometric model of a polycrystal (although the same arguments apply in E^3). Each cell may be occupied by a differently oriented crystal, with all the crystals belonging to any specific crystal class. The latter include:

- transverse isotropy modelling, say, sedimentary rocks at long wavelengths;
- tetragonal modelling, say, wulfenite ($PbMoO_4$);
- trigonal modelling, say, dolomite ($CaMg(CO_3)_2$);
- orthotropic modelling, say, wood;
- triclinic modelling, say, microcline feldspar.

Thus, we need to be able to model fourth-rank tensor random fields, point-wise taking values in any crystal class. While the crystal orientations from grain to grain are random, they are not spatially independent of each other – the assignment of crystal properties over the tessellation is not white noise. This is precisely where the two-point characterisation of the random field of elasticity tensor is needed. While the simplest correlation structure to admit would be white noise, a (much) more realistic model would account for any mathematically admissible correlation structures as dictated by the statistically wide-sense homogeneous and isotropic assumption. A specific correlation can then be fitted to physical measurements.

Note that it may also be of interest to work with a mesoscale random continuum approximation defined by placing a mesoscale window at any spatial position, as shown in Figure 0.1(b). Clearly, the larger the mesoscale window, the weaker the random fluctuations in the mesoscale elasticity tensor: this is the trend to homogenise the material when upscaling from a statistical volume element (SVE) to a representative volume element (RVE). A simple paradigm of this upscaling, albeit only in terms of a scalar random field, is the opacity of a sheet of paper held against light: the further away the sheet is from our eyes, the more homogeneous it appears. Similarly, in the case of upscaling of elastic properties, on any finite scale there is almost certainly anisotropy, and this anisotropy, with mesoscale increasing, tends to zero hand-in-hand with the fluctuations, and it is in the infinite mesoscale limit (i.e. RVE) that material isotropy is obtained as a consequence of the statistical isotropy.

Another motivation for the development of TRF models is to have a realistic input of elasticity random fields into stochastic field equations such as stochastic partial differential equations (SPDE) and stochastic finite elements (SFE). The classical paradigm of SPDE can be written in terms of the anti-plane elastostatics (with $u \equiv u_3$):

$$\boldsymbol{\nabla} \cdot (C(\boldsymbol{x}, \omega) \nabla u) = 0, \quad \boldsymbol{x} \in \mathbb{E}^2, \quad \omega \in \Omega, \tag{0.3}$$

with $C(\cdot, \omega)$ being spatial realisations of a scalar RF. In view of the foregoing discussion, Equation (0.3) is well justified for a piecewise-constant description of realisations of a random medium such as a multiphase composite made of locally isotropic grains. However, in the case of a boundary value problem set up on coarser (i.e. mesoscale) scales, having continuous realisations of properties, a second-rank tensor random field (TRF) of material properties would be much more appropriate: see Figure 0.1(b). The field equation should then read

$$\boldsymbol{\nabla} \cdot (\boldsymbol{C}(\boldsymbol{x}, \omega) \cdot \boldsymbol{\nabla} u) = 0, \qquad \boldsymbol{x} \in E^2, \quad \omega \in \Omega,$$

where $\boldsymbol{C}$ is the second-rank tensor random field.

Moving to the in-plane or 3D elasticity, the starting point is the *Navier equation* of motion (written in symbolic and tensor notations):

$$\mu \nabla^2 \boldsymbol{u} + (\lambda + \mu) \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{u}) = \rho \ddot{\boldsymbol{u}} \quad \text{or} \quad \mu u_{i,jj} + (\lambda + \mu) u_{j,ji} = \rho \ddot{u}_i. \tag{0.4}$$

Here $\boldsymbol{u}$ is the displacement field, λ and μ are two Lamé constants and ρ is the mass density. This equation is often (e.g. in stochastic wave propagation) used as an Ansatz, typically with the pair (λ, μ) taken *ad hoc* as a 'vector' random field with some simple correlation structure for both components. However, in order to properly introduce the smooth randomness in λ and μ, one has to go one step back in derivation of (0.4) and write

$$\mu \nabla^2 \boldsymbol{u} + (\lambda + \mu) \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \boldsymbol{\nabla} \mu \left(\nabla \boldsymbol{u} + (\boldsymbol{\nabla} \boldsymbol{u})^\top \right) + \boldsymbol{\nabla} \lambda \boldsymbol{\nabla} \cdot \boldsymbol{u} = \rho \ddot{\boldsymbol{u}},$$

or

$$\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} + \mu_{,j} (u_{j,i} + u_{i,j}) + \lambda_{,i} u_{j,j} = \rho \ddot{u}_i. \tag{0.5}$$

While two extra terms are now correctly present on the left-hand side, this equation still suffers from the drawback (just as did Equation (0.3)) of local isotropy so that, again by micromechanics upscaling arguments, should be replaced by

$$\boldsymbol{\nabla}\cdot(\boldsymbol{C}\boldsymbol{\nabla}\cdot\boldsymbol{u})^\top=\rho\ddot{\boldsymbol{u}}\quad\text{or}\quad\left(C_{ijkl}u_{(k,l)}\right)_{,j}=\rho\ddot{u}_i.\tag{0.6}$$

Here $\boldsymbol{C}$ $(=C_{ijkl}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l)$, which, at any scale finitely larger than the microstructural scale, is almost surely (a.s.) anisotropic. Clearly, instead of Equation (0.5) one should work with this SPDE (0.6) for $\boldsymbol{u}$.

The foregoing arguments motivate the main goal of this book: to obtain explicit representations of correlation functions of TRFs of ranks 1 through 4, so as to enable their simulation and the construction of models of various field phenomena, subject to the restrictions imposed by the field equations dictated by physics. Briefly, in the case of dependent TRFs, we have, say, the linear momentum equation restricting the Cauchy stress or the angular momentum equation restricting the Cauchy and couple stresses. In the case of material property fields (elasticity, diffusion, permeability . . .) there are conditions of positiveness of either the energy density or the entropy production, as the case may be. In turn, any such conditions lead to restrictions on the respective correlation functions. An introduction to a wide range of continuum physics theories where tensor random fields are needed is given in Chapter 1.

What Mathematical Background is Required?

Random functions of more than one real variable, or *random fields*, appeared for the very first time in applied physical papers. We would like to mention papers by Friedmann & Keller (1924), von Kármán (1937), von Kármán & Howarth (1938), Kampé de Fériet (1939), Obukhov (1941*a*), Obukhov (1941*b*), Robertson (1940), Yaglom (1948), Yaglom (1957), Lomakin (1964) and Lomakin (1965). The physical models introduced in the above papers follow the same scheme, which we explain below. The mathematical tools we use are described in detail in Chapter 2; see also Olive & Auffray (2013) and Auffray, Kolev & Petitot (2014).

Let $(E,\mathbb{R}^d,+)$ be the d-dimensional affine space. The underlying linear space $V=\mathbb{R}^d$ consists of vectors $\boldsymbol{x}=(x_1,\dots,x_d)^\top$. We are mainly interested in the case of $d=2$, which corresponds to *plane problems of continuum physics* as well as in the case of $d=3$ that corresponds to *space problems*. Let $(\cdot,\cdot)$ be the standard inner product in $\mathbb{R}^d$:

$$(\boldsymbol{x},\boldsymbol{y})=\sum_{i=1}^d x_iy_i.$$

Let r be a non-negative integer. The above inner product induces inner products in the space $V^{\otimes r}$ as follows: $(\alpha,\beta)=\alpha\beta$ when $r=0$ and α, $\beta\in V^{\otimes 0}=\mathbb{R}^1$ and

$$(\boldsymbol{S},\boldsymbol{T})=\sum_{j_1=1}^d\cdots\sum_{j_r=1}^d\boldsymbol{S}_{j_1\cdots j_r}\boldsymbol{T}_{j_1\cdots j_r}.$$

The linear transformations of the space $\mathbb{R}^d$ that preserve the above inner product, constitute the *orthogonal group* $\mathrm{O}(d)$. The pair $(g^{\otimes r},V^{\otimes r})$ is an orthogonal representation of the group $\mathrm{O}(d)$ (trivial when $r=0$). Let V_0 be an invariant subspace of the above representation of positive dimension. Let ρ be the restriction of the representation $g \mapsto g^{\otimes r}$ of the group $\mathrm{O}(d)$ to the subspace V_0. Consider the representation (ρ, V_0) as a group action. There are finitely many, say N, orbit types for this action. Let $[G_0]$, …, $[G_{N-1}]$ be the corresponding conjugacy classes of the closed subgroups of the group $\mathrm{O}(d)$. Physicists call them *symmetry classes*. The representatives of conjugacy classes are *point groups*.

Let G_i be a representative of the conjugacy class $[G_i]$. Let V be the subspace of V_0 where the isotypic component of the representation ρ_0 that corresponds to the trivial representation of the group G_i acts. Let $N(G_i)$ be the normaliser of the group G_i in $\mathrm{O}(d)$. Let G be a subgroup of $N(G_i)$ such that G_i is a subgroup of G. Call G the *symmetry group* of a physical material, or the group of material symmetries. The space V is an invariant subspace for the representation $(g^{\otimes r}, (\mathbb{R}^d)^{\otimes r})$ of the group G. Let ρ be the restriction of the above representation to V.

Let $\mathcal{B}$ be a material body that occupies a subset $D \subset E^d$. Consider a physical property of $\mathcal{B}$ that is described by a mapping $\boldsymbol{T}\colon D \to \mathsf{V}$. Examples are given in Subsection 3.1 and include the temperature, where $\mathsf{V} = \mathbb{R}^1$, the velocity of a turbulent fluid, where $\mathsf{V} = \mathbb{R}^d$, the strain tensor of a deformable body, where $\mathsf{V} = \mathsf{S}^2(\mathbb{R}^d)$, the space of symmetric matrices, and the elasticity (or stiffness) tensor, where $\mathsf{V} = \mathsf{S}^2(\mathsf{S}^2(\mathbb{R}^d))$.

To randomise this model, consider a random field $\boldsymbol{T}\colon E \to \mathsf{V}$. Assume that $\mathsf{E}[\|\boldsymbol{T}(A)\|^2] < \infty$, $A \in E$. Assume also that the field $\boldsymbol{T}(A)$ is *mean-square continuous*, that is,

$$\lim_{\|B-A\|\to 0} \mathsf{E}[\|\boldsymbol{T}(B) - \boldsymbol{T}(A)\|^2] = 0$$

for all $A \in E$. Under the translation, the *one-point correlation tensor*

$$\langle \boldsymbol{T}(A) \rangle = \mathsf{E}[\boldsymbol{T}(A)]$$

and the *two-point correlation tensor*

$$\langle \boldsymbol{T}(A), \boldsymbol{T}(B) \rangle = \mathsf{E}[(\boldsymbol{T}(A) - \langle \boldsymbol{T}(A) \rangle) \otimes (\boldsymbol{T}(B) - \langle \boldsymbol{T}(B) \rangle)]$$

do not change. Such a field is called *wide-sense homogeneous*.

Fix a place $O \in D$. Under the rotation of the body about O by a material symmetry $g \in G$, an arbitrary place $A \in D$ becomes the place $O + g(A - O)$. Evidently, the tensor $\boldsymbol{T}(A)$ becomes the tensor $\rho(g)\boldsymbol{T}(A)$. The one-point correlation tensor of the transformed field must be equal to that of the original field:

$$\langle \boldsymbol{T}(O + g(A-O)) \rangle = \rho(g) \langle \boldsymbol{T}(A) \rangle.$$

The two-point correlation tensors of both fields must be equal as well:

$$\langle \boldsymbol{T}(O + g(A-O)), \boldsymbol{T}(O + g(B-O)) \rangle = (\rho \otimes \rho)(g) \langle \boldsymbol{T}(A), \boldsymbol{T}(B) \rangle.$$

A random field that satisfies the two last conditions is called *wide-sense isotropic*. In what follows we omit the words 'wide-sense'.

The main mathematical problem that is solved in this book is as follows. We would like to *find the general form of the one-point and two-point correlation tensors of a homogeneous and isotropic tensor-valued random field* $\boldsymbol{T}(A)$ *as well as the spectral expansion of the above field*.

To explain what we mean, consider the simplest example. Let ρ be the trivial representation of the symmetry group $G = \mathrm{O}(d)$, and let $\tau(A)$ be the corresponding homogeneous and isotropic random field. Schoenberg (1938) proved that the equation

$$\langle \tau(A), \tau(B) \rangle = 2^{(d-2)/2} \Gamma(d/2) \int_0^\infty \frac{J_{(d-2)/2}(\lambda \|B - A\|)}{(\lambda \|B - A\|)^{(d-2)/2}} \, \mathrm{d}\Phi(\lambda)$$

establishes a one-to-one correspondence between the set of two-point correlation functions of homogeneous and isotropic random fields on the space E and the set of finite Borel measures Φ on $[0, \infty)$. Here Γ denotes the gamma function and J denotes the Bessel function of the first kind.

The paper by Schoenberg (1938) was not mentioned before. The reason is that this paper does not treat random fields at all. Instead, the problem of description of all continuous positive-definite functions $B(\|\boldsymbol{y} - \boldsymbol{x}\|)$ with $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ is considered. Thus, there exists a link between the theory of random fields and the theory of positive-definite functions.

The result by Schoenberg (1938) does not help to perform a computer simulation of sample paths of a homogeneous and isotropic random field. The following result is useful for the above purposes. Yaglom (1961) and M. Ĭ. Yadrenko, in his unpublished PhD thesis, proved that a homogeneous and isotropic random field has the following spectral expansion:

$$\tau(A - O) = \langle \tau(A) \rangle + \sqrt{2^{d-1} \Gamma(d/2) \pi^{d/2}} \sum_{\ell=0}^{\infty} \sum_{m=1}^{h(d,\ell)} S_\ell^m(\theta_1, \dots, \theta_{d-2}, \varphi) \times \int_0^\infty \frac{J_{\ell+(d-2)/2}(\lambda\rho)}{(\lambda\rho)^{(d-2)/2}} \, \mathrm{d}Z_\ell^m(\lambda),$$

where $(\rho, \theta_1, \dots, \theta_{d-2}, \varphi)$ are the spherical coordinates of the vector $A - O$, S_ℓ^m are real-valued spherical harmonics and Z_ℓ^m is a sequence of uncorrelated real-valued orthogonal stochastic measures on $[0, \infty)$ with the measure Φ as their common control measure. To simulate the field, we truncate the integrals and use an arbitrary quadrature formula in combination with Monte Carlo simulation.

As the reader can see, the spectral expansion of the field includes an arbitrary choice of the place $O \in E$. There is nothing strange here, because the affine space E does not contain any distinguished places. More explanation is given in Section 2.9. To avoid frequent repetitions of the same words, we vectorise the affine space E by a choice of the origin $O \in E$ once and forever, and denote the vector space E_O by $\mathbb{R}^d$.

The next interesting case is when $\rho(g) = g$. Robertson (1940) proved that in this case the two-point correlation matrix of the field has the form

$$\langle \boldsymbol{v}(\boldsymbol{x}), \boldsymbol{v}(\boldsymbol{y}) \rangle_{ij} = A(\|\boldsymbol{z}\|) z_i z_j + B(\|\boldsymbol{z}\|)\delta_{ij},$$

where $\boldsymbol{z} = \boldsymbol{y} - \boldsymbol{x}$. Note that δ_{ij} is the only covariant tensor of degree 0 and of order 2 of the group O(3), while $z_i z_j$ is its only covariant tensor of degree 2 and of order 2. Thus, another link has been established, this time between the theory of random fields and the classical invariant theory. A review of the invariant theory is given in Section 2.7.

In Section 3.1 we continue to describe the results obtained by our predecessors. As the reader will see, the list of results is impressively short. The complete solution to the problem formulated above requires a combination of tools from different areas of mathematics. No book that describes all necessary tools in a short form is known to the authors. Therefore, in Chapter 2 we collected all of them together. The choice of material was dictated by the solution strategy, and we describe it below.

The main idea is quite simple; see Malyarenko (2013). We *describe the set of homogeneous random fields and reject those that are not isotropic*. Trying this way, we immediately meet the first obstacle: there exist no complete description of two-point correlation tensors of homogeneous random fields taking values in a *real* finite-dimensional linear space. The only known result is as follows. Let $\tilde{\mathsf{V}}$ be a *complex* finite-dimensional linear space. The equation

$$\langle \boldsymbol{T}(\boldsymbol{x}), \boldsymbol{T}(\boldsymbol{y}) \rangle = \int_{\hat{V}} \mathrm{e}^{\mathrm{i}(\boldsymbol{p}, \boldsymbol{y}-\boldsymbol{x})} \, \mathrm{d}F(\boldsymbol{p})$$

establishes a one-to-one correspondence between the set of $\tilde{\mathsf{V}}$-valued homogeneous random fields on the *space domain* V and the set of measures F on the Borel σ-field $\mathfrak{B}(\hat{V})$ taking values in the set of Hermitian non-negative-definite operators on $\tilde{\mathsf{V}}$. Here $\hat{V}$ denote the *wavenumber domain*.

Now we have to define a *real* subspace V of the complex space $\tilde{\mathsf{V}}$. The easiest way to do that is to introduce coordinates in $\tilde{\mathsf{V}}$. We do not want to proceed this way, however, for the following reason. The formulae that describe the solution are basis-dependent. Therefore, the choice of the most convenient basis is a part of the proof. The idea is to make the above choice at the latest possible stage of proof: that is, to write as many formulae as possible in a coordinate-free form.

To start with, we introduce a real structure J in the space $\tilde{\mathsf{V}}$. The eigenvectors of J that correspond to the eigenvalue 1, form a *real* linear space V. The linear space of all Hermitian operators on $\tilde{\mathsf{V}}$ is isomorphic to $\mathsf{V}\otimes\mathsf{V} = \mathsf{S}^2(\mathsf{V})\oplus\Lambda^2(\mathsf{V})$. Let $\top$ be the linear operator in $\mathsf{V}\otimes\mathsf{V}$ for which $\mathsf{S}^2(\mathsf{V})$ is the set of eigenvectors with eigenvalue 1, and $\Lambda^2(\mathsf{V})$ is the set of eigenvectors with eigenvalue -1 (this is just the coordinate-free definition of the transposed matrix). We have the following *necessary condition*: if a homogeneous random field takes values in V, then the measure F satisfies the *reality condition*:

$$F(-A)=F(A)^{\top},\qquad A\in\mathfrak{B}(\hat{V}),$$

where $-A=\{-\boldsymbol{T}\colon\boldsymbol{T}\in A\}$. If one rejects away all Radon measures F that do not satisfy the above condition, no V-valued homogeneous random fields are lost (but some $\tilde{\mathsf{V}}$-valued fields may still remain).

The above method dictates the content of Section 2.1, where we explain many results of linear and tensor algebra in both coordinate and coordinate-free form.

Next, we prove that the one-point correlation tensor of an isotropic random field is a tensor lying in the isotypic subspace of the representation ρ that corresponds to its trivial component, while the measure F must satisfy the condition

$$F(gA)=(\rho\otimes\rho)(g)F(A),\qquad A\in\mathfrak{B}(\hat{V}).$$

The next idea is as follows. We find a group $\tilde{G}$ and its orthogonal representation $(\tilde{\rho},\tilde{\mathsf{V}})$ in a real finite-dimensional space $\tilde{\mathsf{V}}$ such that the above condition *and* the reality condition together are equivalent to the condition

$$F(A)\in\tilde{\mathsf{V}},\qquad F(\tilde{g}A)=\tilde{\rho}(\tilde{g})F(A).$$

Lemma 1 solves this problem. Proof of Lemma 1 requires both general knowledge of group representations and specific knowledge of orthogonal representations, that are given in Section 2.5.

The next step is to introduce the measure $\mu(A)=\operatorname{tr}F(A)$, $A\in\mathfrak{B}(\hat{V})$, noting that F is absolutely continuous with respect to μ, and to write the two-point correlation tensor of the field as

$$\langle\boldsymbol{T}(\boldsymbol{x}),\boldsymbol{T}(\boldsymbol{y})\rangle=\int_{\hat{V}}\mathrm{e}^{\mathrm{i}(\boldsymbol{p},\boldsymbol{y}-\boldsymbol{x})}f(\boldsymbol{p})\,\mathrm{d}\mu(\boldsymbol{p}),$$

where the density $f(\boldsymbol{p})$ is a measurable function on $\hat{V}$ taking values in the *convex compact* set of all Hermitian non-negative-definite operators on $\tilde{\mathsf{V}}$ with unit trace. The measure μ and the density $f(\boldsymbol{p})$ must satisfy the following conditions:

$$\mu(\tilde{g}A)=\mu(A),\qquad f(\tilde{g}\boldsymbol{p})=\tilde{\rho}(\tilde{g})f(\boldsymbol{p}).$$

The description of all possible measures μ is well known. It includes a detailed description of the *stratification* of the space $\hat{V}$ induced by the group action of the group G by the matrix-vector multiplication. In particular, the measure μ is uniquely determined by a Radon measure Φ on the Borel σ-field of the orbit space $\hat{V}/\tilde{G}$. All necessary tools from topology are presented in Section 2.2.

To find all measurable functions $f\colon\hat{V}\to\tilde{\mathsf{V}}$ satisfying the second condition, we proceed as follows. Let $[\tilde{G}_0]$, …, $[\tilde{G}_{M-1}]$ be the symmetry classes of the representation $(g,\hat{V})$ of the group $\tilde{G}$, where $[\tilde{G}_0]$ is the minimal symmetry class, and $[\tilde{G}_{M-1}]$ is the principal symmetry class. Let $(\hat{V}/\tilde{G})_0$, …, $(\hat{V}/\tilde{G})_{M-1}$ be the corresponding stratification of the orbit space $\hat{V}/\tilde{G}$. For simplicity of notation, assume that there is a chart $\boldsymbol{\lambda}_m$ of the manifold $(\hat{V}/\tilde{G})_m$ that covers a dense subset of the above manifold, and there is a chart $\boldsymbol{\varphi}_m$ of the orbit $\tilde{G}/H_m$ that covers a dense subset of the orbit. Let $(\boldsymbol{\lambda}_m, \boldsymbol{\varphi}^0_m)$ be the coordinates of the intersection of the orbit $\tilde{G}\cdot\boldsymbol{\lambda}_m$ with the set $(\hat{V}/\tilde{G})_m$. We have $\tilde{g}(\boldsymbol{\lambda}_m, \boldsymbol{\varphi}^0_m) = (\boldsymbol{\lambda}_m, \boldsymbol{\varphi}^0_m)$ for all $\tilde{g} \in H_m$. It follows that

$$f(\boldsymbol{\lambda}_m, \boldsymbol{\varphi}^0_m) = \tilde{\rho}(\tilde{g}) f(\boldsymbol{\lambda}_0, \boldsymbol{\varphi}^0_m), \qquad \tilde{g} \in H_m.$$

In other words, the tensor $f(\boldsymbol{\lambda}_m, \boldsymbol{\varphi}^0_m)$ lies in the isotypic subspace W_m of the trivial component of the representation $\tilde{\rho}$ of the group H_m. The intersection of the space W_m and the convex compact set of Hermitian non-negative-definite operators in $\tilde{\mathsf{V}}$ is a convex compact set, say $\mathcal{C}_m$. No other restriction exists; that is, the restriction of f to $(\hat{V}/\tilde{G})_m$ is an arbitrary measurable function taking values in $\mathcal{C}_m$.

Now we introduce coordinates. The space V consists of tensors of rank r. Let $\boldsymbol{T}^1_{i_1\cdots i_r}, \dots, \boldsymbol{T}^{\dim \mathsf{V}}_{i_1\cdots i_r}$ be an orthonormal basis in V. The space $\mathsf{V}\otimes\mathsf{V}$ can be represented as the direct sum of the subspace of symmetric tensors and the subspace of skew-symmetric tensors over V:

$$\mathsf{V}\otimes\mathsf{V} = \mathsf{S}^2(\mathsf{V}) \oplus \Lambda^2(\mathsf{V}).$$

Put $\tau(\boldsymbol{T}^1 \oplus \boldsymbol{T}^2) = \boldsymbol{T}^1 \oplus \mathrm{i}\boldsymbol{T}^2$, where $\boldsymbol{T}^1 \in \mathsf{S}^2(\mathsf{V})$, $\boldsymbol{T}^2 \in \Lambda^2(\mathsf{V})$. The map τ is an isomorphism between $\mathsf{V}\otimes\mathsf{V}$ and the real linear space H of Hermitian operators on $\tilde{\mathsf{V}}$. The *coupled basis* of the space H is formed by the tensors

$$\tau(\boldsymbol{T}^i_{i_1\cdots i_r}) \otimes \tau(\boldsymbol{T}^j_{j_1\cdots j_r}), \qquad 1 \le i, j \le \dim \mathsf{V},$$

while the mth *uncoupled basis* of the above space consists of the rank $2r$ tensors

$$\boldsymbol{T}^{0k}_{i_1\cdots j_r}, \qquad 1 \le k \le (\dim \mathsf{V})^2,$$

where the first $\dim \mathsf{W}_m$ tensors constitute an orthonormal basis in W_m.

Let $(\boldsymbol{\lambda}_m, \boldsymbol{\varphi}^0_m)$ be the coordinates of the intersection of the orbit $\tilde{G}\cdot\boldsymbol{\lambda}_m$ with the set $(\hat{V}/\tilde{G})_m$. Let $f^k_{i_1\cdots j_r}(\boldsymbol{\lambda}_m, \boldsymbol{\varphi}^0_m)$ be the value of the linear form $f(\boldsymbol{\lambda}_0, \boldsymbol{\varphi}^0_m)$ on the basis tensor $\boldsymbol{T}^{0k}_{i_1\cdots j_r}$. Then we have $f^k_{i_1\cdots j_r}(\boldsymbol{\lambda}_0, \boldsymbol{\varphi}^0_m) = 0$ when $k > \dim \mathsf{W}_0$. The value of the linear form $f(\boldsymbol{\lambda}_m, \boldsymbol{\varphi}_m)$ on the above basis tensor is then

$$f^k_{i_1\cdots j_r}(\boldsymbol{\lambda}_m, \boldsymbol{\varphi}_m) = \sum_{l=1}^{\dim \mathsf{W}_0} \tilde{\rho}^0_{kl}(\boldsymbol{\varphi}_m) f^l_{i_1\cdots j_r}(\boldsymbol{\lambda}_m, \boldsymbol{\varphi}_m),$$

where $\tilde{\rho}^0_{kl}(\boldsymbol{\varphi}_m) = (\tilde{\rho}(\tilde{g})\boldsymbol{T}^{0k}_{i_1\cdots j_r}, \boldsymbol{T}^{0l}_{i_1\cdots j_r})$ are the matrix entries of the operator $\tilde{\rho}(\tilde{g})$ in the zeroth uncoupled basis, with $\tilde{g}$ being an arbitrary element of $\tilde{G}$ that transforms the point $\boldsymbol{\lambda}_m \in (\hat{V}/\tilde{G})_m$ to the point $(\boldsymbol{\lambda}_m, \boldsymbol{\varphi}_m) \in \hat{V}$.

The tensors of the coupled basis are linear combinations of the tensors of the zeroth uncoupled basis:

$$\tau(\boldsymbol{T}^i_{i_1\cdots i_r}) \otimes \tau(\boldsymbol{T}^j_{j_1\cdots j_r}) = \sum_{k=1}^{(\dim \mathsf{W})^2} c^{mk}_{ij} \boldsymbol{T}^{0k}_{i_1\cdots j_r},$$