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1

Approximation of Univariate Functions

1.1 Introduction

The primary problem in approximation theory is the choice of a successful method of approximation. In this chapter and in Chapter 2 we test various approaches, based on the concept of width, to the evaluation of the quality of a method of approximation. We take as an example the approximation of periodic functions of a single variable. The two main parameters of a method of approximation are its accuracy and complexity. These concepts may be treated in various ways depending on the particular problems involved. Here we start from classical ideas about the approximation of functions by polynomials. After Fourier's 1807 article the representation of a 2π-periodic function by its Fourier series became natural. In other words, the function $f(x)$ is approximately represented by a partial sum $S_n(f,x)$ of its Fourier series:

$$S_n(f,x) := a_0/2 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx),$$
$$a_k := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx, \qquad b_k := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx.$$

We are interested in the approximation of a function f by a polynomial $S_n(f)$ in some L_p-norm, $1 \le p \le \infty$. In the case $p = \infty$ we assume that we are dealing with the uniform norm. As a measure of the accuracy of the method of approximating a periodic function by means of its Fourier partial sum we consider the quantity $\|f - S(f)\|_p$. The complexity of this method of approximation contains the following two characteristics. The order of the trigonometric polynomial $S_n(f)$ is the quantitative characteristic. The following observation gives us the qualitative characteristic. The coefficients of this polynomial are found by the Fourier formulas, which means that the operator S_n is the orthogonal projector onto the subspace of trigonometric polynomials of order n.

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In 1854 Chebyshev suggested representing continuous function f by its polynomial of best approximation, namely, by the polynomial $t_n(f)$ such that

$$\|f - t_n(f)\|_\infty = E_n(f)_\infty := \inf_{\alpha_k,\beta_k} \left\| f(x) - \sum_{k=0}^{n} (\alpha_k \cos kx + \beta_k \sin kx) \right\|_\infty .$$

He proved the existence and uniqueness of such a polynomial. We consider this method of approximation not only in the uniform norm but in all L_p-norms, $1 \le p < \infty$. The accuracy of the Chebyshev method can be easily compared with the accuracy of the Fourier method:

$$E_n(f)_p \le \left\| f - S_n(f) \right\|_p .$$

However, it is difficult to compare the complexities of these two methods. The quantitative characteristics coincide but the qualitative characteristics are different (for example, it is not difficult to understand that for $p = \infty$ the mapping $f \to t_n(f)$ is not a linear operator). The Du Bois–Reymond 1873 example of a continuous function f such that $\left\| f - S_n(f) \right\|_\infty \to \infty$ when $n \to \infty$, and the Weierstrass theorem which says that for each continuous function f we have $E_n(f)_\infty \to 0$ as $n \to \infty$, showed the advantage of the Chebyshev method over the Fourier method from the point of view of accuracy.

The desire to construct methods of approximation which have the advantages of both the Fourier and Chebyshev methods has led to the study of various methods of summation of Fourier series. The most important among them from the point of view of approximation are the de la Vallée Poussin, Fejér, and Jackson methods, which were constructed early in the twentieth century. All these methods are linear. For example, in the de la Vallée Poussin method a function f is approximated by the polynomial

$$V_n(f) := \frac{1}{n} \sum_{l=n}^{2n-1} S_l(f)$$

of order $2n - 1$.

From the point of view of accuracy this method is close to the Chebyshev method; de la Vallée Poussin proved that

$$\left\| f - V_n(f) \right\|_p \le 4E_n(f)_p, \qquad 1 \le p \le \infty.$$

From the point of view of complexity it is close to the Fourier method, and the property of linearity essentially distinguishes it from the Chebyshev method.

We see that common to all these methods is approximation by trigonometric polynomials. However, the methods of constructing these polynomials differ: some

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methods use orthogonal projections on to the subspace of trigonometric polynomials of fixed order, some use best-approximation operators, and some use linear operators.

Thus, the approximation of periodic functions by trigonometric polynomials is natural and this problem has been thoroughly studied. The approximation of functions by algebraic polynomials has been studied in parallel with approximation by trigonometric polynomials. We now point out some results, which determined the style of investigation of a number of problems in approximation theory. These problems are of interest even today.

It was proved by de la Vallée Poussin (1908) that, for best approximation of the function $|x|$ in the uniform norm on $[-1,1]$ by algebraic polynomials of degree n, the following upper *estimate* or *bound* holds:

$$e_n\big(|x|\big) \le C/n.$$

He raised the question of the possibility of an improvement of this estimate in the sense of order. In other words, could the function C/n be replaced by a function that decays faster to zero? Bernstein (1912) proved that this order estimate is sharp. Moreover, he then established the asymptotic behavior of the sequence $\big\{e_n\big(|x|\big)\big\}$ (see Bernstein, 1914):

$$e_n\big(|x|\big) = \mu/n + o(1/n), \qquad \mu = 0.282 \pm 0.004.$$

These results initiated a series of investigations into best approximations of individual functions having special singularities.

At this stage of investigation the natural conjecture arose that the smoother a function, the more rapidly its sequence of best approximations decreases.

In 1911 Jackson proved the inequality

$$E_n(f)_\infty \le Cn^{-r}\omega(f^{(r)}, 1/n)_\infty.$$

The relations which give upper estimates for the best approximations of a function in terms of its smoothness are now called the *Jackson inequalities*, and in a wider sense such relations are called *direct theorems* of approximation theory.

As a result of Bernstein's (1912) and de la Vallée Poussin's (1908, 1919) investigations we can formulate the following assertion, which is now called the *inverse theorem* of approximation theory. If

$$E_n(f)_\infty \le Cn^{-r-\alpha}, \qquad 0 \le r \text{ integer}, \qquad 0 < \alpha < 1,$$

then f has a continuous derivative of order r which belongs to the class Lip α; that is, $f \in W^r H^\alpha$ (in the notation of this book it is the class $H_\infty^{r+\alpha}$). Thus, the results of Jackson, Bernstein, and de la Vallée Poussin show that functions from the class $W^r H^\alpha$, $0 < \alpha < 1$, can be characterized by the order of decrease of its sequences of best approximations.

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We remark that at that time, early in the twentieth century, classes similar to W^rH^α were used in other areas of mathematics for obtaining the orders of decrease of various quantities. As an example we formulate a result of Fredholm (1903). Let $f(x,y)$ be continuous on $[a,b]\times[a,b]$ and

$$\max_{x,y}\big|f(x,y+t)-f(x,y)\big|\le C|t|^\alpha, \qquad 0<\alpha\le 1.$$

Then for eigenvalues $\lambda(J_f)$ of the integral operator

$$(J_f\psi)(x)=\int_a^b f(x,y)\psi(y)dy$$

the following relation is valid for any $\rho>2/(2\alpha+1)$:

$$\sum_{n=1}^{\infty}\big|\lambda_n(J_f)\big|^\rho<\infty.$$

The investigation of the upper bounds or estimates of errors of approximation of functions from a fixed class by some method of approximation began with an article by Lebesgue (1910). In particular, Lebesgue proved that

$$S_n(\mathrm{Lip}\,\alpha)_\infty:=\sup_{f\in\mathrm{Lip}\,\alpha}\big\|f-S_n(f)\big\|_\infty\asymp n^{-\alpha}\ln n.$$

Here and later we write $a_n\asymp b_n$ for two sequences a_n and b_n if there are two positive constants C_1 and C_2 such that $C_1b_n\le a_n\le C_2b_n$ for all n.

The problem of approximation of functions in the classes W^rH^α by trigonometric polynomials was so natural that a tendency to find either asymptotic or exact values of the following quantities appeared:

$$S_n(W^rH^\alpha)_\infty:=\sup_{f\in W^rH^\alpha}\|f-S_n(f)\|_\infty, \qquad E_n(W^rH^\alpha)_\infty:=\sup_{f\in W^rH^\alpha}E_n(f)_\infty.$$

We now formulate the first results in this direction. Kolmogorov (1936) proved the relation (in our notation $W^r=W^r_{\infty,r}$, see §1.4)

$$S_n(W^r)_\infty=\frac{4}{\pi^2}\frac{\ln n}{n^r}+O(n^{-r}), \qquad n\to\infty.$$

Independently, Favard (1937) and Akhiezer and Krein (1937) proved the equality

$$E_n(W^r)_\infty=K_r(n+1)^{-r},$$

where K_r is a number depending on the natural number r.

In 1936 Kolmogorov introduced the concept of the width d_n of a class F in a space X:

$$d_n(F,X):=\inf_{\{\phi_j\}_{j=1}^n}\sup_{f\in F}\inf_{\{c_j\}_{j=1}^n}\left\|f-\sum_{j=1}^n c_j\phi_j\right\|_X.$$

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This concept allows us to find, for a fixed n and for a class F, a subspace of dimension n that is optimal with respect to the construction of a best approximating element. In other words, the concept of width allows us to choose from among various Chebyshev methods having the same quantitative characteristic of complexity (the dimension of the approximating subspace) the one which has the greatest accuracy.

The first result about widths (Kolmogorov, 1936), namely

$$d_{2n+1}(W_2^r, L_2) = (n+1)^{-r},$$

showed that the best subspace of dimension $2n+1$ for the approximation of classes of periodic functions is the subspace of trigonometric polynomials of order n. This result confirmed that the approximation of functions in the class W_2^r by trigonometric polynomials is natural. Further estimates of the widths $d_{2n+1}(W_{q,\alpha}^r, L_p)$, $1 \le q$, $p \le \infty$, some of which are discussed in §2.1 below, showed that, for some values of the parameters $q,\ p$, the subspace of trigonometric polynomials of order n is optimal (in the sense of the order of decay) but for other values of q, p this subspace is not optimal.

The Ismagilov (1974) estimate for the quantity $d_n(W_1^r, L_\infty)$ gave the first example, where the subspace of trigonometric polynomials of order n is not optimal. This phenomenon was thoroughly studied by Kashin (1977).

In analogy to the problem of the *Kolmogorov width*, that is, to the problem concerning the best Chebyshev method, problems concerning the best linear method and the best Fourier method were considered.

Tikhomirov (1960b) introduced the *linear width*:

$$\lambda_n(F, L_p) := \inf_{A:\operatorname{rank} A \le n} \sup_{f \in F} \|f - Af\|_p,$$

and Temlyakov (1982a) introduced the *orthowidth* (Fourier width):

$$\varphi_n(F, L_p) := \inf_{\text{orthonormal system } \{u_i\}_{i=1}^n} \sup_{f \in F} \left\| f - \sum_{i=1}^n \langle f, u_i \rangle u_i \right\|_p.$$

A discussion and comparison of results concerning $d_n(W_q^r, L_p)$, $\lambda_n(W_q^r, L_p)$ and $\varphi_n(W_q^r, L_p)$ can be found in §2.1. Here we remark that, from the point of view of the orthowidth, the Fourier operator S_n is optimal (in the sense of order) for all $1 \le q$, $p \le \infty$ with the exception of the two cases $(1,1)$ and (∞,∞).

Keeping in mind the primary question about the selection of an optimal subspace of approximating functions, we now draw some conclusions from this brief historical survey.

(1) The trigonometric polynomials have been considered as a natural means of approximation of periodic functions during the whole period of development of approximation theory.

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(2) In approximation theory (as well as in other fields of mathematics) it has turned out that it is natural to unite functions with the same smoothness into a class.
(3) The subspace of trigonometric polynomials has been obtained in many cases as the solution of problems regarding the most precise method for the classes of smooth functions: the Chebyshev method (which uses the Kolmogorov width), the linear method (which uses the linear width), or the Fourier method (which uses the orthowidth).

On the basis of these remarks we may formulate the following general strategy for investigating approximation problems; we remark that this strategy turns out to be most fruitful in those cases where we do not know *a priori* a natural method of approximation. First, we solve the width problem for a class of interest in the simplest case, that of approximation in Hilbert space, L_2. Second, we study the system of functions obtained and apply it to approximation in other spaces L_p. This strategy will be used in Chapters 3, 4, and 5.

1.2 Trigonometric Polynomials

Functions of the form

$$t(x) = \sum_{|k|\le n} c_k e^{ikx} = a_0/2 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

(c_k, a_k, b_k are complex numbers) will be called trigonometric polynomials of order n. We denote the set of such polynomials by $\mathscr{T}(n)$ and the subset of $\mathscr{T}(n)$ of real polynomials by $\mathscr{R}\mathscr{T}(n)$.

We first consider a number of concrete polynomials that play an important role in approximation theory.

1.2.1 The Dirichlet Kernel of Order n

The classical univariate Dirichlet kernel of order n is defined as follows:

$$\begin{aligned}\mathscr{D}_n(x) &:= \sum_{|k|\le n} e^{ikx} = e^{-inx}(e^{i(2n+1)x}-1)(e^{ix}-1)^{-1}\\ &= \frac{\sin(n+1/2)x}{\sin(x/2)}. \end{aligned} \tag{1.2.1}$$

The Dirichlet kernel is an even trigonometric polynomial with the majorant

$$\big|\mathscr{D}_n(x)\big| \le \min\big(2n+1, \pi/|x|\big), \qquad |x|\le\pi. \tag{1.2.2}$$

The estimate

$$\|\mathscr{D}_n\|_1 \le C\ln n, \qquad n = 2,3,\dots, \tag{1.2.3}$$

follows from (1.2.2).

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We mention the well-known relation (see Dzyadyk, 1977, p. 112)

$$\|\mathscr{D}_n\|_1 = \frac{4}{\pi^2}\ln n + R_n, \qquad |R_n| \le 3, \qquad n = 1,2,3,\dots$$

For any trigonometric polynomial $t \in \mathscr{T}(n)$ we have

$$\mathscr{D}_n * t := (2\pi)^{-1}\int_{\mathbb{T}} \mathscr{D}_n(x-y)t(y)dy = t.$$

Denote

$$x^l := 2\pi l/(2n+1), \qquad l = 0,1,\dots,2n.$$

Clearly, the points x^l, $l = 1,\dots,2n$, are zeros of the Dirichlet kernel $\mathscr{D}_n$ on $[0,2\pi]$. For any $|k| \le n$ we have

$$\sum_{l=0}^{2n} e^{ikx^l}\mathscr{D}_n(x-x^l) = \sum_{|m|\le n} e^{imx}\sum_{l=0}^{2n} e^{i(k-m)x^l} = e^{ikx}(2n+1).$$

Consequently, for any $t \in \mathscr{T}(n)$,

$$t(x) = (2n+1)^{-1}\sum_{l=0}^{2n} t(x^l)\mathscr{D}_n(x-x^l). \tag{1.2.4}$$

Further, it is easy to see that for any $u,\ v \in \mathscr{T}(n)$ we have

$$\langle u,v\rangle := (2\pi)^{-1}\int_{-\pi}^{\pi} u(x)\overline{v(x)}dx = (2n+1)^{-1}\sum_{l=0}^{2n} u(x^l)\overline{v(x^l)} \tag{1.2.5}$$

and, for any $t \in \mathscr{T}(n)$,

$$\|t\|_2^2 = (2n+1)^{-1}\sum_{l=0}^{2n}\left|t(x^l)\right|^2. \tag{1.2.6}$$

For $1 < q \le \infty$ the estimate

$$\|\mathscr{D}_n\|_q \le C(q)n^{1-1/q} \tag{1.2.7}$$

follows from (1.2.2). Applying the Hölder inequality (see (A.1.1) in the Appendix) for estimating $\|\mathscr{D}_n\|_2^2$ we get

$$2n+1 = \|\mathscr{D}_n\|_2^2 \le \|\mathscr{D}_n\|_q\|\mathscr{D}_n\|_{q'}. \tag{1.2.8}$$

Relations (1.2.7) and (1.2.8) imply for $1 < q < \infty$ the relation

$$\|\mathscr{D}_n\|_q \asymp n^{1-1/q}. \tag{1.2.9}$$

Relation (1.2.9) for $q = \infty$ is obvious.

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We denote by S_n the operator taking a partial sum of order n. Then for $f \in L_1$ we have

$$S_n(f) := \mathscr{D}_n * f = (2\pi)^{-1} \int_{-\pi}^{\pi} \mathscr{D}_n(x-y) f(y) dy.$$

Theorem 1.2.1 *The operator S_n does not change polynomials from $\mathscr{T}(n)$ and for $p = 1$ or ∞ we have*

$$\|S_n\|_{p\to p} \le C \ln n, \qquad n = 2, 3, \dots,$$

and for $1 < p < \infty$ for all n we have

$$\|S_n\|_{p\to p} \le C(p).$$

This theorem follows from (1.2.3) and the Marcinkiewicz multiplier theorem (see Theorem A.3.6).

For $t \in \mathscr{T}(n)$,

$$t(x) = a_0/2 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx),$$

we call the polynomial $\tilde{t} \in \mathscr{T}(n)$, where

$$\tilde{t}(x) := \sum_{k=1}^{n} (a_k \sin kx - b_k \cos kx)$$

the polynomial conjugate to t.

Corollary 1.2.2 *For $1 < p < \infty$ and all n we have*

$$\|\tilde{t}\|_p \le C(p) \|t\|_p.$$

Proof Let $t \in \mathscr{T}(n)$. It is not difficult to see that $\tilde{t} = t * \widetilde{\mathscr{D}}_n$, where

$$\widetilde{\mathscr{D}}_n(x) := 2 \sum_{k=1}^{n} \sin kx.$$

Clearly, it suffices to consider the case of odd n. Let this be the case and set $m := (n+1)/2$, $l := (n-1)/2$. Representing $\widetilde{\mathscr{D}}_n(x)$ in the form

$$\widetilde{\mathscr{D}}_n(x) = \frac{1}{i} \left(\sum_{k=1}^{n} e^{ikx} - \sum_{k=-n}^{-1} e^{ikx} \right) = \frac{1}{i} \left(e^{imx} \mathscr{D}_l(x) - e^{-imx} \mathscr{D}_l(x) \right),$$

we obtain the corollary. □

A trigonometric conjugate operator maps a function $f(x)$ to a function

$$\sum_k (\operatorname{sign} k) \hat{f}(k) e^{ikx}.$$

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The Marcinkiewicz multiplier theorem A.3.6 implies that this operator is bounded as an operator from L_p to L_p for $1 < p < \infty$. We denote by $\tilde{f}$ the conjugate function.

1.2.2 The Fejér Kernel of Order $n-1$

The classical univariate Fejér kernel of order $n-1$ is defined as follows:

$$\mathcal{K}_{n-1}(x) := n^{-1} \sum_{m=0}^{n-1} \mathcal{D}_m(x) = \sum_{|m| \le n} \big(1 - |m|/n\big) e^{imx}$$
$$= \frac{\big(\sin(nx/2)\big)^2}{n\big(\sin(x/2)\big)^2}.$$

The Fejér kernel is an even nonnegative trigonometric polynomial in $\mathcal{T}(n-1)$ with majorant

$$\big|\mathcal{K}_{n-1}(x)\big| = \mathcal{K}_{n-1}(x) \le \min\big(n, \pi^2/(nx^2)\big), \qquad |x| \le \pi. \tag{1.2.10}$$

From the obvious relations

$$\|\mathcal{K}_{n-1}\|_1 = 1, \qquad \|\mathcal{K}_{n-1}\|_\infty = n$$

and the inequality, see (A.1.6),

$$\|f\|_q \le \|f\|_1^{1/q} \|f\|_\infty^{1-1/q}$$

we get in the same way as we obtained (1.2.9),

$$Cn^{1-1/q} \le \|\mathcal{K}_{n-1}\|_q \le n^{1-1/q}, \qquad 1 \le q \le \infty. \tag{1.2.11}$$

1.2.3 The de la Vallée Poussin Kernels

The classical univariate de la Vallée Poussin kernel with parameters m, n is defined as follows:

$$\mathcal{V}_{m,n}(x) := (n-m)^{-1} \sum_{l=m}^{n-1} \mathcal{D}_l(x), \qquad n > m.$$

It is convenient to represent this kernel in terms of Fejér kernels:

$$\mathcal{V}_{m,n}(x) = (n-m)^{-1}\big(n\mathcal{K}_{n-1}(x) - m\mathcal{K}_{m-1}(x)\big)$$
$$= (\cos mx - \cos nx)\Big(2(n-m)\big(\sin(x/2)\big)^2\Big)^{-1}.$$

The de la Vallée Poussin kernels $\mathcal{V}_{m,n}$ are even trigonometric polynomials of order $n-1$ with majorant

$$\big|\mathcal{V}_{m,n}(x)\big| \le C \min\Big(n,\ 1/|x|, 1/\big((n-m)x^2\big)\Big), \qquad |x| \le \pi. \tag{1.2.12}$$

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Relation (1.2.12) implies the estimate

$$\|\mathscr{V}_{m,n}\|_1 \le C\ln\big(1+n/(n-m)\big).$$

We often use the de la Vallée Poussin kernel with $n = 2m$ and denote it by

$$\mathscr{V}_m(x) := \mathscr{V}_{m,2m}(x), \qquad m \ge 1, \qquad \mathscr{V}_0(x) := 1.$$

Then for $m \ge 1$ we have

$$\mathscr{V}_m = 2\mathscr{K}_{2m-1} - \mathscr{K}_{m-1},$$

which, with the properties of $\mathscr{K}_n$, implies

$$\|\mathscr{V}_m\|_1 \le 3. \tag{1.2.13}$$

In addition,

$$\|\mathscr{V}_m\|_\infty \le 3m.$$

Consequently, in the same way as above, see (1.2.9) and (1.2.11), we get

$$\|\mathscr{V}_m\|_q \asymp m^{1-1/q}, \qquad 1 \le q \le \infty. \tag{1.2.14}$$

Denote

$$x(l) := \pi l/2m, \qquad l = 1,\dots,4m.$$

Then, analogously to (1.2.4), for each $t \in \mathscr{T}(m)$ we have

$$t(x) = (4m)^{-1}\sum_{l=1}^{4m} t\big(x(l)\big)\mathscr{V}_m\big(x-x(l)\big). \tag{1.2.15}$$

The operator $\mathscr{V}_m$ defined on L_1 by the formula

$$V_m(f) := f * \mathscr{V}_m$$

is called the de la Vallée Poussin operator.

The following theorem is a corollary of the definition of the kernels $\mathscr{V}_m$ and the relation (1.2.13).

Theorem 1.2.3 *The operator V_m does not change polynomials from $\mathscr{T}(m)$, and for all $1 \le p \le \infty$ we have*

$$\|V_m\|_{p\to p} \le 3, \qquad m = 1,2,\dots$$

In addition, we formulate two properties of the de la Vallée Poussin kernels.

(1) Relation (1.2.12) with $n = 2m$ implies the inequality

$$\big|\mathscr{V}_m(x)\big| \le C\min\big(m, 1/(mx^2)\big), \qquad |x| \le \pi.$$

It is easy to derive from this inequality the following property.

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