Part I

Review of General Relativity

Cambridge University Press
978-1-108-42874-3 — Extensions of f(R) Gravity
Tiberiu Harko , Francisco S. N. Lobo
Excerpt
More Information

1

Introduction

Einstein's GR is one of the greatest intellectual achievements ever conceived by the human mind. In General Relativity (GR) the basic assumption is that any gravitational field can be interpreted geometrically, and it is directly related to a significant variation of the space-time metric $g_{\mu\nu}$. Geometrically, the metric tensor provides the infinitesimal distance between two neighboring points of the space-time continuum. Therefore, in GR the gravitational field is fully determined by the quantities that describe the intrinsic geometrical properties and structure of the space-time. This important idea has the fundamental implication that the space-time geometry itself (such as its metric and its curvature) is locally induced by physical phenomena and processes (generally involving the mass distribution or the motion of gravitating objects), and that space and time are not a priori determined absolute concepts. For an arbitrary gravitational field, which generally varies in both space and time, the metric of the four-dimensional space-time is *non-Euclidean* (Riemannian). Therefore, its geometric properties cannot be described any longer by the simple and well known results of Euclidean geometry, which is constructed based on Euclid's fifth postulate of the parallels, which dictates that through an arbitrary point one can construct one and only one parallel to a given straight line.

The fifth postulate of the parallels, as formulated by Euclid, is a fundamental assertion in today's "elementary" geometry, and it cannot be reduced to a more basic axiom, or proven independently. It clearly differentiates the Euclidean space, and its underlying geometry, from other mathematical spaces that could be constructed from different geometrical considerations. The first examples of non-Euclidean spaces were discovered by János Bolyai (1802–1860) and Nikolai Lobachevski (1793–1856), and belong to what is called today hyperbolic geometry. A different class of geometries – the spherical geometries – was found by Georg Bernhard Riemann (1826–1866). All these early works were based on, and greatly influenced by, the profound geometric research of Karl Friederich Gauss (1777–1855). Presently all these different classes of geometric theories, based on a number of common axioms and principles, are called Riemannian geometry. One of the most remarkable evolutions in modern physics was the natural emergence of the ideas of the Riemannian geometry as introduced in the framework of gravitational fields, which took place in the years 1909–1916, and is due to Albert Einstein (1879–1955) and his coworker Marcel Grossmann (1878–1936).

Note that the earlier Newtonian model of the universe is based on the fundamental concepts of absolute space, obeying the axioms and postulates of Euclidean geometry, and of absolute time, dominated by a static gravitational force proportional to the inverse of the square of the distance. Thus, Newton's model is superseded, in Einstein's theory, by a four-dimensional complex space-time geometry where the rules of the Euclidean geometry do not hold any longer. The curved trajectories, which Newtonian gravity imposes upon free-falling bodies in the Euclidean–Newtonian representation, are interpreted in GR by the lines of minimal length in the curved space-time of the Riemannian–Einsteinian approach.

In order to describe physical phenomena we must first fix a reference frame, which can be chosen arbitrarily. But a fundamental requirement in physics is that the laws of nature must be written in such a way that they are valid in any four-dimensional system of coordinates, independently of the coordinates we are using. This fundamental statement can also be formulated through the requirement that all laws of nature must be written in a *covariant form*, making them independent of the reference frame. Of course, this general requirement does not allude to the physical equivalence of all possible systems of coordinates or reference frames (as in the case of the special theory of relativity, where all inertial frames of reference are indeed equivalent). Contrary to Special Relativity, in GR the specific occurrences of physical processes, encompassing the properties of the motion of bodies in different physical fields, vary in different systems of reference. Thus, the first step in building the theory of gravitation is to construct the mathematical framework that would allow the formulation of the laws of physics in a covariant manner. This is carried out in Chapter 2.

GR is based, from a physical point of view, on the equivalence principle, which states the equivalence of the non-inertial frames of reference with certain gravitational fields. However, similarly to Newtonian gravitational physics, in GR there is a fundamental difference between *real gravitational fields*, and *fields that are generated by the motion of non-inertial reference systems*. The most important property of the "true" gravitational fields is that no coordinate transformation can cancel them. Therefore, once a gravitational field is present, the geometry of the space-time has the fundamental property that the elements of the metric tensor $g_{\mu\nu}$ cannot be brought by any admissible transformation of coordinates to their constant Galilean values, over all space-time, with the metric tensor reduced to a diagonal form. Space-times in which the metric tensor cannot be reduced globally to a diagonal form with constant values of the components are called *curved*, in contrast to the case of the flat (Euclidian/Galilean) space-times, where such a transformation can always be performed.

However, by a specific choice of the coordinate system we can transform all the metric tensor components $g_{\mu\nu}$ to a diagonal Galilean form at any fixed point of the curved (Riemannian) space-time. From a mathematical point of view this amounts to the transformation to a diagonal form of a quadratic form characterized by constant coefficients (the components of the metric tensor $g_{\mu\nu}$ at the given point, having a pseudo-Euclidian signature). After performing, at the given point, the transformation of the local metric tensor to the diagonal form, in our sign convention the matrix of the metric tensor components will have one negative and three positive principal values. We will call this configuration of signs the *signature* of the metric tensor, and of the associated matrix. From this mathematical result it follows immediately that the *determinant*, given by $g = \det |g_{\mu\nu}|$, constructed from the elements of the matrix of the metric tensor $g_{\mu\nu}$, is always *negative* for a physically acceptable space-time, so that $g < 0$.

The basic idea of GR is that once a gravitational field is present in a given region of the universe, the intrinsic geometry of the space-time is non-Euclidian (Riemannian). This is indeed the case for both true gravitational fields, whose properties are fully determined by the curvature of the space-time, as well as for fields described by a non-inertial reference frame. In the geometric description of gravity proposed by GR, the assignment of the system of coordinates to a given frame, in which the natural processes are described, is not restricted in any way, be it mathematical or physical. The set of the three space coordinates (x^1, x^2, x^3) can be chosen arbitrarily as quantities describing the position of physical objects in the ordinary space. The time coordinate x^0 can be introduced with the help of a clock running in an arbitrary way.

In order to describe the gravitational field in the framework of GR, the Einstein gravitational field equations are defined in a Riemannian geometry, and they establish a deep connection between the geometric properties of the space-time and its matter content, providing a full description of both geometric characteristics of the space-time, and of the dynamics. In Chapter 3, we will derive the Einstein field equations, and discuss their mathematical properties, as well as some of their physical implications.

Thus, GR provides an excellent description of the gravitational effects and phenomena, in particular, for the weak field limit within the boundaries of the solar system. More than 150 years ago, astronomical observations pointed out an anomaly in the motion of Mercury, the innermost planet, which seemed to defy the Newtonian laws of gravitation. The planetary orbits as derived from Newton's inverse square law are stable ellipses, while the orbit of Mercury presents a precession of its perihelion, which could not be explained by using the then "standard" Newtonian laws of motion. However, the precession can be fully explained once we adopt the geometric description of GR, according to which the planets move on geodesic orbits in the curved Riemannian space-time generated by the presence of the Sun. Not only do planets feel the change in geometry, but light, and more generally electromagnetic waves, are also affected by the curvature of the space-time when traveling in the vicinity of the Sun. The presence of the curvature induces an observable deflection of the light rays, and a similarly detectable time delay in the propagation of radar signals.

GR also has a profound implication on the properties of compact objects, like stars, whose interior geometry is very different from that of their Newtonian counterparts. Once a star reaches three solar masses it will collapse to form a new and strange object called a black hole, whose intriguing properties can be well understood and explained in the framework of Einstein's theory of GR. Black holes can absorb matter in an astrophysical process called accretion, and the radiation emitted by the accreted matter represents an important astrophysical evidence for the possible existence of black holes in the universe. In Chapter 4, we will review the fundamental observational tests of GR that have been carried out at the solar system level and discuss some of the basic properties of compact, stellar type astrophysical objects, such as the description of accretion processes.

Turning now to large scales, cosmology is the scientific (physical/mathematical) study of the universe, of its components, and of its history. The major questions one may ask in cosmology are: How did the universe come into being? How did it evolve in time, and what is its fate? As a science, cosmology has evolved tremendously in the past 20 years, with unexpected (and revolutionary) new information about the origin, structure, and evolution of the universe coming in at a high rate. This information was mostly acquired through recent and significant technological improvements in telescope designs, and from space missions. Today cosmology has basically become a search for the decoding of not only what composes of the universe (the astrophysical objects within it, and their material composition), but also of its overall architecture, and past and present history. The beginning of modern cosmology can be traced back to the observational and theoretical advances made in the early twentieth century. In those times astronomers generally adopted the view that our galaxy (the Milky Way) had the shape of a disk, and was an isolated object in an infinite universe. However, there were many visible celestial objects, called spiral nebulae, such as M31 (the Andromeda galaxy), whose positioning with respect to our galaxy were not precisely known. Were these nebulae located inside the Milky Way, or far away from it?

In 1912, Vesto Slipher (1875–1969) investigated the electromagnetic spectra from the spiral nebulae, and found that many of them were Doppler-shifted. This means that the frequency of the light emitted by these nebulae was influenced by the speed of the source (in the same way as the frequency of sound changes for a passing train). Then astronomers quickly became aware that not only were the spiral nebulae (galaxies) moving rapidly away from our galaxy, in which the Earth is located, but they were shifting away from each other as well. Hence, once this information became available, astronomers started to interpret these galactic

Cambridge University Press
978-1-108-42874-3 — Extensions of f(R) Gravity
Tiberiu Harko , Francisco S. N. Lobo
Excerpt
More Information

motions in terms of a global expansion of the universe. By 1925, Slipher had investigated around 40 galaxies, and found that electromagnetic spectra displaying redshifted lines were much more abundant than those exhibiting blueshifted spectra. Thus, Slipher realized that almost all the galaxies he investigated were quickly moving away from our galaxy, although Andromeda's galaxy blueshifted spectral lines indicated that it was approaching our Milky Way galaxy at a speed of approximately 300 km/s.

A key advance in astronomy occurred in 1918, when powerful methods to measure the distances to the spiral nebulae (today known to be independent galaxies) were developed. To determine astronomical distances Harlow Shapley (1885–1972) introduced the use of Cepheids, bright stars whose periodic pulsations indicated periods ranging between a minimum of a few days to a maximum of one month. The period of the variability of the luminosity of these stars is very precisely related to their absolute luminosity, which can be calibrated with a high precision by using the known distance of the neighboring Large Magellanic Cloud, a satellite galaxy of the Milky Way, located at a distance of 50 kiloparsecs from it. During the years between 1923 and 1929, Edwin Hubble (1889–1953) was able to detect, with the use of the 100-inch" telescope at Mount Wilson in California, the Cepheid stars in M31. Hubble advanced a new astronomical distance measurement technique, by employing the observational data of the brightest stars in the more remote galaxies. Hubble supplemented his galactic distance determinations with Slipher's redshift data of the spiral nebulae (galaxies), to make one of the most amazing discoveries in the history of science, namely, that *all galaxies are receding from us*. Moreover, he also found the law describing the universal expansion, which is given by a simple proportionality relation relating the velocity v of the galaxy and its distance d to the Earth or, more exactly, by Hubble's law of galactic expansion $v = H_0 d$. In Hubble's expansion law of the universe the constant of proportionality H_0 is denoted today *Hubble's constant*, and it has units of km/s/Mpc. Its precise determination is one of the central issues of present day observational cosmology.

In 1917, Einstein proposed the first general relativistic cosmological model, corresponding to a static, homogeneous, and isotropic universe, having a spherical geometry [183]. This model raised a number of interesting theoretical questions. The gravitational pull of matter led to an instability (acceleration) in this model, something Einstein did not expect, and did not want on observational grounds, since at that time the expansion of the universe was still to be discovered. Thus, Einstein modified his equations for GR by introducing a new term, proportional to the metric tensor, with the constant scalar proportionality coefficient called *the cosmological constant*, and denoted by Λ. This new term counteracts the gravitational attraction of matter, and hence it can be interpreted as describing *a kind of antigravity effect*. But even after the introduction of the cosmological constant, it turns out that the Einstein static universe is still not stable against small perturbations. The first to investigate the geometrical and physical

Cambridge University Press
978-1-108-42874-3 — Extensions of f(R) Gravity
Tiberiu Harko , Francisco S. N. Lobo
Excerpt
More Information

properties of Einstein's static universe was the astronomer Willem de Sitter (1872–1934) [165, 166]. By using the Einstein equations with the cosmological constant, and by adopting a different mathematical model, de Sitter obtained a new solution of the field equations for a vacuum universe with vanishing energy density and pressure. However, the de Sitter space is not static, and by means of a transformation of the coordinates [502], the de Sitter metric can be reformulated into a dynamical representation, which still plays an essential role in modern cosmology.

In the early 1920s, the Russian mathematician and meteorologist Alexander Friedmann (1889–1925) realized that Einstein's gravitational field equations have non-static solutions that could set out an expanding universe [196], whose size is a function of time. Friedmann's solutions showed that our universe was born in one single event [306], about 13 thousand million years ago. Even today all the galaxies are still traveling apart from us due of this initial "explosion." Hence the Friedmann solutions imply that all matter, the universe itself, and space and time themselves appeared at once, in a single instant. The British astronomer Fred Hoyle (1915–2001), proponent and strong supporter of an alternative cosmological theory called the *steady state theory*, derogatorily labeled this model as a "big bang," and under this name it became the standard cosmological paradigm of our times, according to which the universe was born in a single point, in a state of very high density and temperature. After learning about Friedmann's work, and the discovery of the expansion of the universe, Einstein immediately discarded the cosmological constant, regarding it as the biggest blunder of his life.

The first major confirmation of the big bang theory came in 1964, when, using a horn antenna (7.35 cm) at Bell Labs, Arno Penzias (1933–) and Robert Wilson (1936–) [411] accidentally detected an isotropic cosmological microwave background, a distant echo that survived up to the present day from the primeval big bang "explosion." The Cosmic Microwave Background radiation is the main observational evidence for the hot big bang model. It has a perfect blackbody spectrum, and its temperature today has been determined to be $T = 2.73$ K (that is, a rather cold $-270°$C).

April 1992 represents another milestone in the history of modern cosmology. The COBE satellite team released the results on the discovery of anisotropies in the Cosmic Microwave Background radiation (CMB) at the level of 10^{-5} K, or one part in 100,000. The CMB temperature measurements provide a snapshot of the primeval matter density fluctuations that eventually led to the formation of galaxies, a process that started when the universe was around $t \approx 400{,}000$ years old. The map of the sky obtained by COBE, and by the next satellite experiments, is also the best evidence for the cosmological principle, claiming that the universe possesses a high degree of isotropy (or spherical symmetry). The research initiated by COBE were extended and significantly improved by another satellite experiment, the Wilkinson Microwave Anisotropy Probe (WMAP). The

Cambridge University Press
978-1-108-42874-3 — Extensions of f(R) Gravity
Tiberiu Harko , Francisco S. N. Lobo
Excerpt
More Information

WMAP team also provided a detailed and significantly improved full-sky map of the oldest detectable electromagnetic radiation in the universe. WMAP recorded and analyzed in detail microwave radiation from 379,000 years after the big bang, that is, around 13 billion years ago. Recently, the Planck satellite mission [11] significantly improved the observations made by WMAP, and provided high precision determinations of several crucial cosmological parameters, including the estimation of the average density of ordinary matter (baryonic density parameter), and of the density parameter of the dark matter in the universe.

If the attractive force of gravity were the only force determining the expansion dynamics of the universe, then we would expect the universe to be quickly decelerating and, in the limiting situation of a universe having essentially zero total energy, having an expansion rate decreasing as $1/t$. That's why the observational discovery of the late-time acceleration of the universe, initially announced in the years 1998–1999 by two distinct research teams (led by Riess and Schmidt [436] and by Perlmutter [413]), astonished most cosmologists and general relativists. Much work has been devoted to this unexpected cosmological effect, and many observational investigations as well as theoretical studies performed in the past 20 years have confirmed this amazing phenomenon. Moreover, a number of other observations, including the Planck satellite data [11], have led to the astonishing result that the observed universe consists only in a proportion of 4–5% of ordinary matter, that is, matter composed of baryons (protons, neutrons, etc.), electrons, and the other known elementary particles. Around 95% of the energy-matter balance of the universe consists of two *basically unknown components*, *dark matter* ($\sim$25%) and *dark energy* ($\sim$70%), respectively. These amazing results have led to the formulation of another paradigm in modern cosmology, the ΛCDM (Λ Cold Dark Matter) paradigm, which assumes (cold) dark matter as the major matter component in a universe whose late-time dynamics is determined by Einstein's cosmological constant, also giving the preponderent contribution to the total energy balance of the universe. In Chapter 5, we will review some of the basic results and models in modern cosmology.

Thus, in this context, modern astrophysical and cosmological models are plagued with two severe theoretical problems, namely, the dark energy and the dark matter enigmas. Relative to the latter, the dynamics of test particles around galaxies, as well as the corresponding mass discrepancy in galactic clusters, is explained by postulating the existence of a hypothetical form of dark matter particle. Relative to the dark energy problem, as mentioned in the previous paragraph, high precision observational data has confirmed with startling evidence that the universe is undergoing a phase of accelerated expansion. This phase is one of the most important and challenging current problems in cosmology, and represents a new imbalance in the governing gravitational equations. Several candidates, responsible for this expansion, have been proposed in the literature, in particular, dark energy models and modified theories of gravity, among others. The simplest scenario to explain the late-time cosmic speedup is to invoke the ΛCDM paradigm. However, if we assume that the cosmological constant constitutes the vacuum energy of the gravitational field, we are faced with an extremely embarrassing discrepancy of 120 orders of magnitude between the observed value and that predicted by quantum field theory. This is the celebrated cosmological constant problem.

The physical motivations for these modifications of GR also consist in the possibility of a more realistic representation of the gravitational fields near curvature singularities, and to create some first order approximation for the quantum theory of gravitational fields. It is clear that these questions involve not only gravity, but also particle physics. String theory provides a synthesis of these two branches of physics and is widely believed to be moving toward a viable quantum gravity theory. However, when adopting string theory as a full theory of quantum gravity, one does not recover GR in the low-energy limit, but rather a scalar–tensor theory of gravity. The initial motivations for scalar–tensor theories arose from the need to implement the Machs principle, which was not fully incorporated in GR. In fact, pioneering renormalization approaches to GR clearly showed the need for the introduction of counterterms, implying the presence of extra degrees of freedom, in addition to the spin two massless gravitons. It was shown that the corrections introduced by renormalization are at least quadratic in the curvature invariants, which lead to extensions of the Einstein–Hilbert Lagrangian.

In Chapter 6, we will consider an extension of the Einstein–Hilbert action, namely, $f(R)$ gravity, which contains several appealing features, as it combines mathematical simplicity and a fair amount of generality. Chapter 6 will also serves as a bridge to the final parts of the book, where we consider an intensive analysis of two extensions to $f(R)$ gravity considered in the literature, namely, modified theories of gravity with couplings between curvature and matter, and hybrid metric-Palatini gravity. While these modified theories of gravity offer an alternative explanation to the standard cosmological model for the expansion history of the universe, it offers a paradigm for nature fundamentally distinct from dark energy models of cosmic acceleration, even those that perfectly mimic the same expansion history.

Thus, a goal of this work is to perform a theoretical and phenomenological analysis of specific infrared modifications of GR, and to find the consistency of the generalized curvature-matter couplings in modified gravity and the hybrid metric-Palatini theory. Finally, one of the expected outcomes and impact of this work is to deepen the theoretical understanding of the dynamics of the universe and the perplexing nature of gravity itself.