CAMBRIDGE TRACTS IN MATHEMATICS

General Editors

212 Applications of Diophantine Approximation to Integral Points and Transcendence

CAMBRIDGE TRACTS IN MATHEMATICS

A complete list of books in the series can be found at www.cambridge.org/mathematics.
Recent titles include the following:

178. Analysis in Positive Characteristic. By A. N. Kochubei
179. Dynamics of Linear Operators. By F. Bayart and É. Matheron
180. Synthetic Geometry of Manifolds. By A. Kock
181. Totally Positive Matrices. By A. Pinkus
182. Nonlinear Markov Processes and Kinetic Equations. By V. N. Kolokoltsov
183. Period Domains over Finite and p-adic Fields. By J.-F. Dat, S. Orlik, and M. Rapoport
184. Algebraic Theories. By J. Adámek, J. Rosický, and E. M. Vitale
185. Rigidity in Higher Rank Abelian Group Actions I: Introduction and Cocycle Problem. By A. Katok and V. Niţică
186. Dimensions, Embeddings, and Attractors. By J. C. Robinson
187. Convexity: An Analytic Viewpoint. By B. Simon
188. Modern Approaches to the Invariant Subspace Problem. By I. Chalendar and J. R. Partington
189. Nonlinear Perron–Frobenius Theory. By B. Lemmens and R. Nussbaum
190. Jordan Structures in Geometry and Analysis. By C.-H. Chu
191. Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion. By H. Osswald
192. Normal Approximations with Malliavin Calculus. By I. Nourdin and G. Peccati
193. Distribution Modulo One and Diophantine Approximation. By Y. Bugeaud
194. Mathematics of Two-Dimensional Turbulence. By S. Kuksin and A. Shirikyan
195. A Universal Construction for Groups Acting Freely on Real Trees. By I. Chiswell and T. Müller
196. The Theory of Hardy's Z-Function. By A. Ivić
197. Induced Representations of Locally Compact Groups. By E. Kaniuth and K. F. Taylor
198. Topics in Critical Point Theory. By K. Perera and M. Schechter
199. Combinatorics of Minuscule Representations. By R. M. Green
200. Singularities of the Minimal Model Program. By J. Kollár
201. Coherence in Three-Dimensional Category Theory. By N. Gurski
202. Canonical Ramsey Theory on Polish Spaces. By V. Kanovei, M. Sabok, and J. Zapletal
203. A Primer on the Dirichlet Space. By O. El-Fallah, K. Kellay, J. Mashreghi, and T. Ransford
204. Group Cohomology and Algebraic Cycles. By B. Totaro
205. Ridge Functions. By A. Pinkus
206. Probability on Real Lie Algebras. By U. Franz and N. Privault
207. Auxiliary Polynomials in Number Theory. By D. Masser
208. Representations of Elementary Abelian p-Groups and Vector Bundles. By D. J. Benson
209. Non-homogeneous Random Walks. By M. Menshikov, S. Popov, and A. Wade
210. Fourier Integrals in Classical Analysis (Second Edition). By C. D. Sogge
211. Eigenvalues, Multiplicities and Graphs. By C. R. Johnson and C. M. Saiago
212. Applications of Diophantine Approximation to Integral Points and Transcendence. By P. Corvaja and U. Zannier

Applications of Diophantine Approximation to Integral Points and Transcendence

PIETRO CORVAJA
Università degli Studi di Udine, Italy

UMBERTO ZANNIER
Scuola Normale Superiore, Pisa

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre,
New Delhi – 110025, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108424943
DOI: 10.1017/9781108348096

First published 2018

Printed in the United Kingdom by Clays, St Ives plc

A catalogue record for this publication is available from the British Library.

ISBN 978-1-108-42494-3 Hardback

Contents

Preface

The present work originates from a short course (14 hours) given by the second author at the University of Pisa during October 2002; it was addressed to graduate students, who did not necessarily have a specific background.

Notes were taken and collected in a short volume [Z5], which is now out of print.

About ten years later, the first author gave another short course at the Mathematical Science Institute of Chennai, India, dealing with similar topics; the notes have recently been published in [Co2].

In the meantime, several new results had been obtained, and it seemed natural to add some material to the first volume, so as to make it more complete. The present authors had worked on several of the applications presented in the old notes, so they decided to write jointly this entirely new edition.

To write an entirely new volume seemed difficult and much more time consuming; therefore we decided to keep much of the former version of the second author's book [Z5], with just some additions. This also prevented the inclusion of highly interesting results obtained by other authors.

As with the former notes, the present work does not require any particular prerequisites; actually, certain basic notions will be recalled, so the general level may be considered fairly elementary. The style is somewhere in between a survey and a detailed account.

In any case, the last two chapters especially contain more recent material.

Roughly speaking, the contents concern certain applications of Diophantine approximation to Diophantine equations. The whole field is, however, far too vast for a (short) course, or even for a general survey. Therefore we have concentrated on a few topics, involving the celebrated subspace theorem of W. M. Schmidt. However, the (difficult) proof of this theorem will not be discussed, let alone the quantitative versions by J.-H. Evertse, H.-P. Schlickewei, and Schmidt, and the geometric formulations due to Faltings and Wüstholz and to Evertse and Ferretti.

Even within these limitations, we have not always given complete details.

The five chapters contain several exercises, proposed both in the course of the main text and in a separate section near the end of each chapter. Those in the latter category, often containing hints at solutions, sometimes convey known results, which are not inserted in full for the sake of brevity. A $*$ is attached to somewhat more involved exercises.

Insofar as the proofs of the theorems are concerned, we have basically followed the original arguments, but naturally sometimes we have introduced (more or less slight) variations. Also, some statements appear for the first time in the literature, especially concerning concrete examples and applications.

Notation and Conventions

The letters $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ will have their usual meanings and $\overline{\mathbb{Q}}$ will denote an algebraic closure of $\mathbb{Q}$. Usually (but not always) the letter k will denote a number field, with ring of integers $\mathscr{O} = \mathscr{O}_k$; further related notation will be introduced or recalled in Section 1.2.2.

If $P \in k[X_1, \ldots, X_n]$ and if σ is an isomorphism of k in some field, P^σ will denote the polynomial obtained by applying σ to the coefficients of P.

For a group G, the set $\{g^d : g \in G\}$ will be denoted by $[d]G$.

By $\mathbb{G}_m^n$ we shall denote the nth power of the *multiplicative algebraic group* $\mathbb{G}_m$, as recalled in Section 2.3.

For a commutative ring R, we shall denote by R^* the (multiplicative) group of invertible elements in R.

The symbols "$\mathbb{A}^n$" and "$\mathbb{P}_n$" will denote respectively affine and projective n-dimensional spaces. The point of $\mathbb{P}_n$ with homogeneous coordinates $x_0, x_1, \ldots, x_n$ will be denoted by $(x_0 : x_1 : \cdots : x_n)$.

For an algebraic variety V, embedded in some affine or projective space, $V(L)$ will denote the set of points of V with coordinates in the field (or ring, or set, if V is affine) L. We have sometimes used in an equivalent way the terminology "point of V" or "vector of V."

By "V/k" we shall mean that V is defined over the field k, i.e., defined by a system of equations with coefficients in k. In that case, $k(V)$ will denote the function field of V over k; if V is affine, $k[V]$ will denote the coordinate ring over k. (Also some further terminology from algebraic geometry will be standard, following, for example, [H].)

Usually, $\mathbf{X}$ will denote a vector of variables $(X_1, \ldots, X_n)$, while $\mathbf{x}$ will represent suitable specializations of $\mathbf{X}$. For a vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we shall put $\mathbf{X}^{\mathbf{a}} := X_1^{a_1} \cdots X_n^{a_n}$.

The symbols O and $\ll$ will have their usual meanings; namely, for real functions f, g of certain variables, expressions like "$f = O(g)$" and "$f \ll g$" will mean that $|f| \leq C \cdot |g|$ for the relevant values of the variables (which will normally be clear from the context), where the *implied constant* C is a positive number dependent only on certain basic data. These data too will be normally clear from the context; if not, notations like "$f \ll_\varepsilon g$" will mean that C may depend also on the parameter ε. By $f \asymp g$, we mean both $f \gg g$ and $f \ll g$.

Concerning the list of references. Whenever the content of certain original papers has been treated exhaustively in some book, we have often cited only the book, with the double aim of directing the reader toward a more ample source and not expanding the already rather long list. Again to avoid lengthening the list of references, we have occasionally omitted some specific relevant reference, provided that it appears in some other item that has been cited.