CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS 168

PERIOD MAPPINGS AND PERIOD DOMAINS

This up-to-date introduction to Griffiths' theory of period maps and period domains focusses on algebraic, group-theoretic and differential geometric aspects. Starting with an explanation of Griffiths' basic theory, the authors go on to introduce spectral sequences and Koszul complexes that are used to derive results about cycles on higher-dimensional algebraic varieties such as the Noether–Lefschetz theorem and Nori's theorem. They explain differential geometric methods, leading up to proofs of Arakelov-type theorems, the theorem of the fixed part and the rigidity theorem. They also use Higgs bundles and harmonic maps to prove the striking result that not all compact quotients of period domains are Kähler.

This thoroughly revised second edition includes a new third part covering important recent developments, in which the group-theoretic approach to Hodge structures is explained, leading to Mumford–Tate groups and their associated domains, the Mumford–Tate varieties and generalizations of Shimura varieties. This viewpoint also leads to a factorization of the period map which has an arithmetic flavor. Higgs bundles reappear in connection with Shimura varieties.

James Carlson is Professor Emeritus at the University of Utah. From 2003 to 2012, he was president of the Clay Mathematics Institute. Most of Carlson's research is in the area of Hodge Theory.

Stefan Müller-Stach is Professor for number theory at Johannes Gutenberg Universität Mainz, Germany. He works in arithmetic and algebraic geometry, focussing on algebraic cycles and Hodge theory, and his recent research interests include period integrals and the history and foundations of mathematics. Recently, he has published monographs on number theory (with J. Piontkowski) and period numbers (with A. Huber), as well as an edition of some works of Richard Dedekind.

Chris Peters is a retired professor from the Université Grenoble Alpes, France and has a research position at the Technical University of Eindhoven. He is widely known for the monographs *Compact Complex Surfaces* (with W. Barth, K. Hulek and A. van de Ven), as well as *Mixed Hodge Structures*, (with J. Steenbrink). He has also written shorter treatises on the motivic aspects of Hodge theory, on motives (with J.P. Murre and J. Nagel), and on applications of Hodge theory in mirror symmetry (with Bertin).

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit: www.cambridge.org/mathematics.

Already published
129 D. Goldfeld & J. Hundley *Automorphic representations and L-functions for the general linear group, I*
130 D. Goldfeld & J. Hundley *Automorphic representations and L-functions for the general linear group, II*
131 D. A. Craven *The theory of fusion systems*
132 J. Väänänen *Models and games*
133 G. Malle & D. Testerman *Linear algebraic groups and finite groups of Lie type*
134 P. Li *Geometric analysis*
135 F. Maggi *Sets of finite perimeter and geometric variational problems*
136 M. Brodmann & R. Y. Sharp *Local cohomology (2nd Edition)*
137 C. Muscalu & W. Schlag *Classical and multilinear harmonic analysis, I*
138 C. Muscalu & W. Schlag *Classical and multilinear harmonic analysis, II*
139 B. Helffer *Spectral theory and its applications*
140 R. Pemantle & M. C. Wilson *Analytic combinatorics in several variables*
141 B. Branner & N. Fagella *Quasiconformal surgery in holomorphic dynamics*
142 R. M. Dudley *Uniform central limit theorems (2nd Edition)*
143 T. Leinster *Basic category theory*
144 I. Arzhantsev, U. Derenthal, J. Hausen & A. Laface *Cox rings*
145 M. Viana *Lectures on Lyapunov exponents*
146 J.-H. Evertse & K. Győry *Unit equations in Diophantine number theory*
147 A. Prasad *Representation theory*
148 S. R. Garcia, J. Mashreghi & W. T. Ross *Introduction to model spaces and their operators*
149 C. Godsil & K. Meagher *Erdős–Ko–Rado theorems: Algebraic approaches*
150 P. Mattila *Fourier analysis and Hausdorff dimension*
151 M. Viana & K. Oliveira *Foundations of ergodic theory*
152 V. I. Paulsen & M. Raghupathi *An introduction to the theory of reproducing kernel Hilbert spaces*
153 R. Beals & R. Wong *Special functions and orthogonal polynomials*
154 V. Jurdjevic *Optimal control and geometry: Integrable systems*
155 G. Pisier *Martingales in Banach spaces*
156 C. T. C. Wall *Differential topology*
157 J. C. Robinson, J. L. Rodrigo & W. Sadowski *The three-dimensional Navier–Stokes equations*
158 D. Huybrechts *Lectures on K3 surfaces*
159 H. Matsumoto & S. Taniguchi *Stochastic analysis*
160 A. Borodin & G. Olshanski *Representations of the infinite symmetric group*
161 P. Webb *Finite group representations for the pure mathematician*
162 C. J. Bishop & Y. Peres *Fractals in probability and analysis*
163 A. Bovier *Gaussian processes on trees*
164 P. Schneider *Galois representations and* (φ, Γ)*-modules*
165 P. Gille & T. Szamuely *Central simple algebras and Galois cohomology (2nd Edition)*
166 D. Li & H. Queffelec *Introduction to Banach spaces, I*
167 D. Li & H. Queffelec *Introduction to Banach spaces, II*
168 J. Carlson, S. Müller-Stach & C. Peters *Period mappings and period domains (2nd Edition)*

Period Mappings and Period Domains

Second Edition

JAMES CARLSON
University of Utah

STEFAN MÜLLER-STACH
Johannes Gutenberg Universität
Mainz, Germany

CHRIS PETERS
Université Grenoble Alpes, France

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

4843/24, 2nd Floor, Ansari Road, Daryaganj, Delhi – 110002, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108422628
DOI: 10.1017/9781316995846

First published 2003
Second edition 2017

Printed in the United Kingdom by Clays, St Ives plc

A catalogue record for this publication is available from the British Library.

ISBN 978-1-108-42262-8 Hardback
ISBN 978-1-316-63956-6 Paperback

Cambridge University Press
978-1-108-42262-8 — Period Mappings and Period Domains
James Carlson , Stefan Müller-Stach , Chris Peters
Frontmatter
More Information

Contents

Cambridge University Press
978-1-108-42262-8 — Period Mappings and Period Domains
James Carlson , Stefan Müller-Stach , Chris Peters
Frontmatter
More Information

Preface to the Second Edition

In the fourteen years since the first edition appeared, ample experience with teaching to graduate students made us realize that a proper understanding of several of the core aspects of period domains needed a lot more explanation than offered in the first edition of this book, especially with regards to the Lie group aspects of period domains.

Consequently, we decided a thorough reworking of the book was called for. In particular Section 4.3, and Chapters 12 and 13 needed revision. The latter two chapters have been rearranged and now contain more, often completely rewritten sections. At the same time relevant newer developments have been inserted at appropriate places. Finally we added a new "Part Four" with additional, more recent topics. This also required an extra Appendix D about Lie groups and algebraic groups.

Let us be more specific about the added material. There is a new Section 5.4 on counterexamples to infinitesimal Torelli. In Chapter 6 the abstract and powerful formalism of derived functors has been added so that for instance the algebraic treatment of the Gauss–Manin connection could be given, as well as a proper treatment of the Leray spectral sequence. In Chapter 13 we have devoted more detail on Higgs bundles and their logarithmic variant. This made it possible to also include some geometric applications at the end of that chapter.

"Part Four" starts with a chapter explaining the by now standard group theoretic formulation of the concept of a Hodge structure. This naturally leads to Mumford–Tate groups and their associated domains. The chapter culminates with a result giving a factorization of the period map which stresses the role of the Mumford–Tate group of a given variation. In Chapter 16 Mumford–Tate domains and their quotients by certain discrete groups, the Mumford–Tate varieties, are considered from a more abstract, axiomatic point of view. In this chapter the relation with the classical Shimura varieties is also explained. In the next and final chapter we study various interesting subvarieties of Mumford–Tate varieties, especially of low dimension.

One word about the prerequisites. Of course, they remain the same (see page xi), but we should mention a couple of more recent books that may serve as a guide to the reader. There are now many introductory books to algebraic geometry and this is not the place to mention all of these. However, Donaldson's book (2011) starts as we do, from Riemann surfaces, and focusses on Hodge theory. So it serves as a particularly adequate introduction; moreover, its scope is broad and leads to some fascinating recent mathematics. Secondly, in the First Edition, we unfortunately failed to mention explicitly Chern's wonderful introduction to complex manifolds (see Chern, 1967), as well as Hartshorne (1987) although both figured in the bibliography. These are not needed to understand the text, but serve to complement it, Chern's book from the differential geometric side; Hartshorne's book from the algebraic side.

We acknowledge support from the University of Mainz, the French CNRS, the Technical University of Eindhoven, as well as the Deutsche Forschungsgemeinschaft (SFB, Transregio 45). Finally we thank Ana Brecan, Ariyan Javanpeykar, Daniel Huybrechts, Ben Moonen, Jan Nagel and Kang Zuo for their remarks on a preliminary version of this second edition.

Preface to the First Edition

What to expect of this book?

Our aim is to give an up to date exposition of the theory of period maps originally introduced by Griffiths. It is mainly intended as a text book for graduate students. However, it should also be of interest to any mathematician wishing to get introduced to those aspects of Hodge theory which are related to Griffiths' theory.

Prerequisites

We assume that the reader has encountered complex or complex algebraic manifolds before. We have in mind familiarity with the concepts from the first chapters of the book by Griffiths and Harris (1978) or from the first half of the book by Forster (1981).

A second prerequisite is some familiarity with algebraic topology. For the fundamental group the reader may consult Forster's book (loc. cit.). Homology and cohomology are at the base of Hodge theory and so the reader should know either simplicial or singular homology and cohomology. A good source for the latter is Greenberg (1967).

Next, some familiarity with basic concepts and ideas from differential geometry such as smooth manifolds, differential forms, connections and characteristic classes is required. Apart from the book by Griffiths and Harris (1978) the reader is invited to consult the monograph by Guillemin and Pollack (1974). To have an idea of what we actually use in the book, we refer to the appendices. We occasionally refer to these in the main body of the book. We particularly recommend the exercises which are meant to provide the techniques necessary to calculate all sorts of invariants for concrete examples in the main text.

Contents of the book

The concept of a period-integral goes back to the nineteenth century; it was introduced by Legendre and Weierstraß for integrals of certain elliptic functions over closed circuits in the dissected complex plane and of course is related to periodic functions like the Weierstraß $\wp$-function. In modern terminology we would say that these integrals describe exactly how the complex structure of an elliptic curve varies. From this point of view the analogous question for higher genus curves becomes apparent and leads to period matrices and Torelli's theorem for curves. We have treated this historical starting point in the first chapter.

Since we introduce the major concepts of the book by means of examples, this chapter can be viewed as a motivation for the rest of the book. Indeed, period mappings and period domains appear in it, as well as several other important notions and ideas such as monodromy of a family, algebraic cycles, the Hodge decomposition and the Hodge conjecture. This chapter is rather long since we also wanted to address several important aspects of the theory that we do not treat in later chapters but nevertheless motivate parts in it. Below we say more about this, but we pause here to point out that the nature of the first chapter makes it possible to use it entirely for a first course on period maps.

For instance, we already introduce mixed Hodge theory in this chapter and explain the geometry behind it, but of course only in the simplest situations. We look at the cohomology of a singular curve on the one hand, and on the other hand we consider the limit mixed Hodge structure on the cohomology for a degenerating family of curves. This second example leads to the *asymptotic* study and becomes technically complicated in higher dimensions and falls beyond the modest scope of our book. Nevertheless it motivates certain results in the rest of the book such as those concerning variations of Hodge structure over the punctured disk (especially the monodromy theorem) which are considered in detail in Chapter 13.

The beautiful topic of Picard–Fuchs equations, treated in relation to a family of elliptic curves, does not come back in later chapters. We certainly could have done this, for instance after our discussion of the periods for families of hypersurfaces in projective space (Section 3.2). Lack of time and space prevented us from doing this. We refer the interested reader to Bertin and Peters (2002) where some calculations are carried out which are significant for important examples occurring in mirror-symmetry and which can be understood after reading the material in the first part.

The remainder of the first part of the book is devoted to fleshing out the ideas presented in this first chapter. Cohomology being essentially the only available invariant, we explain in Chapter 2 how the Kähler assumption implies that one can pass from the type decomposition on the level of complex forms to the level of cohomology classes. This is the Hodge decomposition. We show how to compute the Hodge decomposition in a host of basic examples. In the next chapter we pave the way for the introduction of the period map by looking at invariants related to cohomology that behave holomorphically (although this is shown much later, in Chapter 6, when we have developed the necessary tools). Griffiths' intermediate Jacobians and the Hodge (p, p)-classes are central in this chapter; we also calculate the Hodge decomposition of the cohomology of projective hypersurfaces in purely algebraic terms. This will enable us on various occasions to use these as examples to illustrate the theory. For instance, infinitesimal Torelli is proved for them in Chapter 5, Noether–Lefschetz type theorems in Chapter 7 and variational Torelli theorems in Chapter 8.

In Chapter 4 the central concepts of this book finally can be defined after we have illustrated the role of the monodromy in the case of Lefschetz pencils. Abstract variations of Hodge structure then are introduced. In a subsequent chapter these are studied from an infinitesimal point of view.

In Part Two spectral sequences are treated and with these, previous loose ends can be tied up. Another central tool, to be developed in Chapter 7 is the theory of Koszul complexes. Through Donagi's symmetrizer lemma and its variants these turn out to be crucial for applications such as Noether–Lefschetz theorems and variational Torelli, treated in Chapter 7 and Chapter 8, respectively.

Then in Chapter 9 we turn to another important ingredient in the study of algebraic cycles, the normal functions. Their infinitesimal study leads to a proof of a by now classical theorem due to Voisin and Green stating that the image of the Abel-Jacobi map for "very general" odd-dimensional hypersurfaces of projective space is as small as it can be, at least if the degree is large enough.

We finish this part with a sophisticated chapter on Nori's theorem which has profound consequences for algebraic cycles, vastly generalizing pioneering results by Griffiths and Clemens.

In Part Three of the book we turn to purely differential geometric aspects of period domains. Our main goal here is to explain in Chapter 13 those curvature properties which are relevant for period maps. Previous to that chapter, in Chapters 11 and 12 we present several more or less well known notions and techniques from differential geometry, which go into the Lie theory needed for period domains.

Among the various important applications of these basic curvature properties we have chosen to prove in Chapter 13 the theorem of the fixed part, the rigidity theorem and the monodromy theorem. We also show that the period map extends as a proper map over the locus where the local monodromy is finite and give

Cambridge University Press
978-1-108-42262-8 — Period Mappings and Period Domains
James Carlson , Stefan Müller-Stach , Chris Peters
Frontmatter
More Information

some important consequences. In the same chapter we introduce Higgs bundles and briefly explain how these come up in Simpson's work on nonabelian Hodge theory.

In Chapter 14 we broaden our point of view in that we look more generally at harmonic and pluriharmonic maps with target a locally symmetric space. Using the results of this study, we can, for instance, show that compact quotients of period domains of even weight are never homotopy equivalent to Kähler manifolds.

To facilitate reading, we start every chapter with a brief outline of its content. To encourage the reader to digest the considerable amount of concepts and techniques we have included many examples and problems. For the more difficult problems we have given hints or references to the literature. Finally, we end every chapter with some historical remarks.

It is our pleasure to thank various people and institutions for their help in the writing of this book.

We are first of all greatly indebted to Phillip Griffiths who inspired us either directly or indirectly over all the years we have been active as mathematicians; through this book we hope to promote some of the exciting ideas and results related to cycles initiated by him and pursued by others, like Herb Clemens, Mark Green, Madhav Nori and Claire Voisin.

Special thanks go to Domingo Toledo for tremendous assistance with the last part of the book and to Jan Nagel who let us present part of his work in Chapter 10. Moreover, he and several others critically read firsts drafts of this book: Daniel Huybrechts, James Lewis, Jacob Murre, Jens Piontkowski and Eckart Viehweg; we extend our gratitude to all of them.

We furthermore acknowledge the support we have received (on various occasions) from the University of Grenoble, the University of Utah at Salt-Lake City and the University of Essen.

Finally we want to thank our respective wives and companions Nicole, Siggi and Annie for their patience and endurance over all the years that it took us to prepare this book.