A First Course in Quantitative Finance

This new and exciting book offers a fresh approach to quantitative finance and utilizes novel new features, including stereoscopic images which permit 3D visualization of complex subjects without the need for additional tools.

Offering an integrated approach to the subject, *A First Course in Quantitative Finance* introduces students to the architecture of complete financial markets before exploring the concepts and models of modern portfolio theory, derivative pricing, and fixed-income products in both complete and incomplete market settings. Subjects are organized throughout in a way that encourages a gradual and parallel learning process of both the economic concepts and their mathematical descriptions, framed by additional perspectives from classical utility theory, financial economics, and behavioral finance.

Suitable for postgraduate students studying courses in quantitative finance, financial engineering, and financial econometrics as part of an economics, finance, econometric, or mathematics program, this book contains all necessary theoretical and mathematical concepts and numerical methods, as well as the necessary programming code for porting algorithms onto a computer.

**Professor Dr. Thomas Mazzoni** has lectured at the University of Hagen and the Dortmund Business School and is now based at the University of Greifswald, Germany, where he received the 2014 award for excellence in teaching and outstanding dedication.
A First Course in Quantitative Finance

THOMAS MAZZONI
University of Greifswald
# Contents

1 Introduction  
   About This Book  
   \hspace{1em}page 1  

Part I Technical Basics  

2 A Primer on Probability  
   2.1 Probability and Measure  
   2.2 Filtrations and the Flow of Information  
   2.3 Conditional Probability and Independence  
   2.4 Random Variables and Stochastic Processes  
   2.5 Moments of Random Variables  
   2.6 Characteristic Function and Fourier-Transform  
   2.7 Further Reading  
   2.8 Problems  
   \hspace{1em}7  

3 Vector Spaces  
   3.1 Real Vector Spaces  
   3.2 Dual Vector Space and Inner Product  
   3.3 Dimensionality, Basis, and Subspaces  
   3.4 Functionals and Operators  
   3.5 Adjoint and Inverse Operators  
   3.6 Eigenvalue Problems  
   3.7 Linear Algebra  
   3.8 Vector Differential Calculus  
   3.9 Multivariate Normal Distribution  
   3.10 Further Reading  
   3.11 Problems  
   \hspace{1em}26  

4 Utility Theory  
   4.1 Lotteries  
   4.2 Preference Relations and Expected Utility  
   4.3 Risk Aversion  
   4.4 Measures of Risk Aversion  
   4.5 Certainty Equivalent and Risk Premium  
   4.6 Classes of Utility Functions  
   \hspace{1em}60
## Contents

#### Part I

4.7 Constrained Optimization 70
4.8 Further Reading 74
4.9 Problems 74

#### Part II

### Financial Markets and Portfolio Theory

5 Architecture of Financial Markets 79
5.1 The Arrow–Debreu-World 79
5.2 The Portfolio Selection Problem 81
5.3 Preference-Free Results 83
5.4 Pareto-Optimal Allocation and the Representative Agent 89
5.5 Market Completeness and Replicating Portfolios 92
5.6 Martingale Measures and Duality 96
5.7 Further Reading 97
5.8 Problems 98

6 Modern Portfolio Theory 100
6.1 The Gaussian Framework 100
6.2 Mean-Variance Analysis 104
6.3 The Minimum Variance Portfolio 109
6.4 Variance Efficient Portfolios 111
6.5 Optimal Portfolios and Diversification 113
6.6 Tobin’s Separation Theorem and the Market Portfolio 115
6.7 Further Reading 118
6.8 Problems 119

7 CAPM and APT 120
7.1 Empirical Problems with MPT 120
7.2 The Capital Asset Pricing Model (CAPM) 121
7.3 Estimating Betas from Market Data 125
7.4 Statistical Issues of Regression Analysis and Inference 129
7.5 The Arbitrage Pricing Theory (APT) 135
7.6 Comparing CAPM and APT 140
7.7 Further Reading 141
7.8 Problems 141

8 Portfolio Performance and Management 143
8.1 Portfolio Performance Statistics 143
8.2 Money Management and Kelly-Criterion 145
8.3 Adjusting for Individual Market Views 150
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.4</td>
<td>Further Reading</td>
<td>154</td>
</tr>
<tr>
<td>8.5</td>
<td>Problems</td>
<td>154</td>
</tr>
<tr>
<td>9</td>
<td>Financial Econcomics</td>
<td>156</td>
</tr>
<tr>
<td>9.1</td>
<td>The Rational Valuation Principle</td>
<td>156</td>
</tr>
<tr>
<td>9.2</td>
<td>Stock Price Bubbles</td>
<td>160</td>
</tr>
<tr>
<td>9.3</td>
<td>Shiller’s Volatility Puzzle</td>
<td>164</td>
</tr>
<tr>
<td>9.4</td>
<td>Stochastic Discount Factor Models</td>
<td>166</td>
</tr>
<tr>
<td>9.5</td>
<td>C-CAPM and Hansen–Jagannathan-Bounds</td>
<td>169</td>
</tr>
<tr>
<td>9.6</td>
<td>The Equity Premium Puzzle</td>
<td>172</td>
</tr>
<tr>
<td>9.7</td>
<td>The Campbell–Cochrane-Model</td>
<td>175</td>
</tr>
<tr>
<td>9.8</td>
<td>Further Reading</td>
<td>179</td>
</tr>
<tr>
<td>9.9</td>
<td>Problems</td>
<td>180</td>
</tr>
<tr>
<td>10</td>
<td>Behavioral Finance</td>
<td>181</td>
</tr>
<tr>
<td>10.1</td>
<td>The Efficient Market Hypothesis</td>
<td>181</td>
</tr>
<tr>
<td>10.2</td>
<td>Beyond Rationality</td>
<td>185</td>
</tr>
<tr>
<td>10.3</td>
<td>Prospect Theory</td>
<td>188</td>
</tr>
<tr>
<td>10.4</td>
<td>Cumulative Prospect Theory (CPT)</td>
<td>191</td>
</tr>
<tr>
<td>10.5</td>
<td>CPT and the Equity Premium Puzzle</td>
<td>194</td>
</tr>
<tr>
<td>10.6</td>
<td>The Price Momentum Effect</td>
<td>197</td>
</tr>
<tr>
<td>10.7</td>
<td>Unifying CPT and Modern Portfolio Theory</td>
<td>199</td>
</tr>
<tr>
<td>10.8</td>
<td>Further Reading</td>
<td>205</td>
</tr>
<tr>
<td>10.9</td>
<td>Problems</td>
<td>205</td>
</tr>
<tr>
<td><strong>Part III</strong></td>
<td><strong>Derivatives</strong></td>
<td>207</td>
</tr>
<tr>
<td>11</td>
<td>Forwards, Futures, and Options</td>
<td>209</td>
</tr>
<tr>
<td>11.1</td>
<td>Forward and Future Contracts</td>
<td>209</td>
</tr>
<tr>
<td>11.2</td>
<td>Bank Account and Forward Price</td>
<td>210</td>
</tr>
<tr>
<td>11.3</td>
<td>Options</td>
<td>213</td>
</tr>
<tr>
<td>11.4</td>
<td>Compound Positions and Option Strategies</td>
<td>216</td>
</tr>
<tr>
<td>11.5</td>
<td>Arbitrage Bounds on Options</td>
<td>219</td>
</tr>
<tr>
<td>11.6</td>
<td>Further Reading</td>
<td>220</td>
</tr>
<tr>
<td>11.7</td>
<td>Problems</td>
<td>220</td>
</tr>
<tr>
<td>12</td>
<td>The Binomial Model</td>
<td>222</td>
</tr>
<tr>
<td>12.1</td>
<td>The Coin Flip Universe</td>
<td>222</td>
</tr>
<tr>
<td>12.2</td>
<td>The Multi-Period Binomial Model</td>
<td>225</td>
</tr>
<tr>
<td>12.3</td>
<td>Valuating a European Call in the Binomial Model</td>
<td>228</td>
</tr>
<tr>
<td>12.4</td>
<td>Backward Valuation and American Options</td>
<td>232</td>
</tr>
</tbody>
</table>
Contents

12.5 Stopping Times and Snell-Envelope 235
12.6 Path Dependent Options 240
12.7 The Black–Scholes-Limit of the Binomial Model 243
12.8 Further Reading 246
12.9 Problems 247

13 The Black–Scholes-Theory 249
13.1 Geometric Brownian Motion and Itô’s Lemma 249
13.2 The Black–Scholes-Equation 253
13.3 Dirac’s δ-Function and Tempered Distributions 256
13.4 The Fundamental Solution 260
13.5 Binary and Plain Vanilla Option Prices 264
13.6 Simple Extensions of the Black–Scholes-Model 267
13.7 Discrete Dividend Payments 270
13.8 American Exercise Right 274
13.9 Discrete Hedging and the Greeks 277
13.10 Transaction Costs 283
13.11 Merton’s Firm Value Model 287
13.12 Further Reading 289
13.13 Problems 290

14 Exotics in the Black–Scholes-Model 291
14.1 Finite Difference Methods 291
14.2 Numerical Valuation and Coding 298
14.3 Weak Path Dependence and Early Exercise 301
14.4 Girsanov’s Theorem 303
14.5 The Feynman–Kac-Formula 306
14.6 Monte Carlo Simulation 310
14.7 Strongly Path Dependent Contracts 312
14.8 Valuating American Contracts with Monte Carlo 317
14.9 Further Reading 323
14.10 Problems 323

15 Deterministic Volatility 326
15.1 The Term Structure of Volatility 326
15.2 GARCH-Models 328
15.3 Duan’s Option Pricing Model 332
15.4 Local Volatility and the Dupire-Equation 334
15.5 Implied Volatility and Most Likely Path 338
15.6 Skew-Based Parametric Representation of the Volatility Surface 343
15.7 Brownian Bridge and GARCH-Parametrization 345
15.8 Further Reading 351
15.9 Problems 351
Contents

16 Stochastic Volatility 353

16.1 The Consequence of Stochastic Volatility 353
16.2 Characteristic Functions and the Generalized Fourier-Transform 355
16.3 The Pricing Formula in Fourier-Space 358
16.4 The Heston–Nandi GARCH-Model 362
16.5 The Heston-Model 365
16.6 Inverting the Fourier-Transform 370
16.7 Implied Volatility in the SABR-Model 373
16.8 Further Reading 377
16.9 Problems 378

17 Processes with Jumps 379

17.1 Càdlàg Processes, Local-, and Semimartingales 379
17.2 Simple and Compound Poisson-Process 381
17.3 GARCH-Models with Conditional Jump Dynamics 385
17.4 Merton's Jump-Diffusion Model 389
17.5 Barrier Options and the Reflection Principle 393
17.6 Lévy-Processes 397
17.7 Subordination of Brownian motion 402
17.8 The Esscher-Transform 406
17.9 Combining Jumps and Stochastic Volatility 410
17.10 Further Reading 412
17.11 Problems 412

Part IV The Fixed-Income World 415

18 Basic Fixed-Income Instruments 417

18.1 Bonds and Forward Rate Agreements 417
18.2 LIBOR and Floating Rate Notes 421
18.3 Day-Count Conventions and Accrued Interest 422
18.4 Yield Measures and Yield Curve Construction 425
18.5 Duration and Convexity 430
18.6 Forward Curve and Bootstrapping 433
18.7 Interest Rate Swaps 436
18.8 Further Reading 440
18.9 Problems 440

19 Plain Vanilla Fixed-Income Derivatives 442

19.1 The T-Forward Measure 442
19.2 The Black-76-Model 445
19.3 Caps and Floors 447
Contents

19.4 Swaptions and the Annuity Measure 449
19.5 Eurodollar Futures 452
19.6 Further Reading 453
19.7 Problems 454

20 Term Structure Models 455

20.1 A Term Structure Toy Model 455
20.2 Yield Curve Fitting 458
20.3 Mean Reversion and the Vasicek-Model 460
20.4 Bond Option Pricing and the Jamshidian-Decomposition 463
20.5 Affine Term Structure Models 465
20.6 The Heath–Jarrow–Morton-Framework 469
20.7 Multi-Factor HJM and Historical Volatility 475
20.8 Further Reading 481
20.9 Problems 481

21 The LIBOR Market Model 483

21.1 The Transition from HJM to Market Models 483
21.2 The Change-of-Numéraire Toolkit 487
21.3 Calibration to Caplet Volatilities 491
21.4 Parametric Correlation Matrices 494
21.5 Calibrating Correlations and the Swap Market Model 498
21.6 Pricing Exotics in the LMM 502
21.7 Further Reading 506
21.8 Problems 506

A Complex Analysis 509

A.1 Introduction to Complex Numbers 509
A.2 Complex Functions and Derivatives 513
A.3 Complex Integration 515
A.4 The Residue Theorem 519

B Solutions to Problems 525

References 573
Index 583