

### **Probability: A Lively Introduction**

Probability has applications in many areas of modern science, not to mention in our daily life, and its importance as a mathematical discipline cannot be overrated. This engaging book, with its easy to follow writing style, provides a comprehensive, yet concise, introduction to the subject. It covers all of the standard material for undergraduate and first-year-graduate-level courses, as well as many topics that are usually not found in standard texts – such as Bayesian inference, Markov chain Monte Carlo simulation, and Chernoff bounds.

The student-friendly text has the following additional features:

- Is the result of many years of teaching and feedback from students
- Stresses why probability is so relevant and how to apply it
- Offers many real-world examples to support the theory
- Includes more than 750 problems with detailed solutions of the odd-numbered problems
- Gives students confidence in their own problem-solving skills

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# Probability: A Lively Introduction

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## Preface

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Why do so many students find probability difficult? Even the most mathematically competent often find probability a subject that is difficult to use and understand. The difficulty stems from the fact that most problems in probability, even ones that are easy to understand, cannot be solved by using cookbook recipes as is sometimes the case in other areas of mathematics. Instead, each new problem often requires imagination and creative thinking. That is why probability is difficult, but also why it is fun and engaging. Probability is a fascinating subject and I hope, in this book, to share my enthusiasm for the subject.

Probability is best taught to beginning students by using motivating examples and problems, and a solution approach that gives the students confidence to solve problems on their own. The examples and problems should be relevant, clear, and instructive. This book is not written in a theorem–proof style, but proofs flow with the subsequent text and no mathematics is introduced without specific examples and applications to motivate the theory. It distinguishes itself from other introductory probability texts by its emphasis on why probability is so relevant and how to apply it. Every attempt has been made to create a student-friendly book and to help students understand what they are learning, not just to learn it.

This textbook is designed for a first course in probability at an undergraduate level or first-year graduate level. It covers all of the standard material for such courses, but it also contains many topics that are usually not found in introductory probability books – such as stochastic simulation. The emphasis throughout the book is on probability, but attention is also given to statistics. In particular, Bayesian inference is discussed at length and illustrated with several illuminating examples. The book can be used in a variety of disciplines, ranging from applied mathematics and statistics to computer science, operations

research, and engineering, and is suitable not only for introductory courses, but also for self-study. The prerequisite knowledge is a basic course in calculus.

Good problems are an essential part of each textbook. The field of probability is well known for being a subject that can best be acquired by the process of learning-by-doing. Much care has been taken to present problems that will enhance the student's understanding of probability. He or she will be asked to think about ideas, rather than simply plugging numbers into formulas. Working through them, it may often be found that probability problems are harder than they first appear. This book has more than 750 carefully designed problems, both "easy ones" and challenging ones. Problems are grouped according to the section they are based on, which should be convenient for both students and instructors. An important feature of this textbook is that it contains detailed solutions to the odd-numbered problems, which helps stimulate active learning and contributes to students' confidence in their own problem-solving skills. It is my belief that there is an enormous increase in content when worked-out solutions to exercises are included. Solutions for the even-numbered problems and further detailed solutions to the odd-numbered problems for instructors can be found on the book's webpage ([www.cambridge.org/TijmsProbability](http://www.cambridge.org/TijmsProbability)). Another added feature is that the student will find many tips on problem-solving strategies.

### **How to Teach from this Book**

This book, which is the result of many years of teaching and feedback from many students, can be used to teach probability courses at several different levels. Since students learn best when they participate actively in the process, the instructor may want to use computer simulation in the course. Appendix D gives the basic ideas of Monte Carlo simulation, together with instructive simulation exercises. Simulation in teaching probability helps to develop and sharpen probabilistic intuition and is popular with many students. Over many years of teaching probability I have also found that the subject of Markov chains is very appealing to students, and the book offers the possibility of a follow-up course on stochastic processes that covers renewal–reward processes, discrete-time Markov chains, and continuous-time Markov chains. In a course on discrete probability, material on discrete-time Markov chains can even be taught right after the notion of conditional probability has been introduced.