Notes on Counting: An Introduction to Enumerative Combinatorics

Enumerative combinatorics, in its algebraic and analytic forms, is vital to many areas of mathematics, from model theory to statistical mechanics. This book, which stems from many years' experience of teaching, invites students into the subject and prepares them for more advanced texts. It is suitable as a class text or for individual study.

The author provides proofs for many of the theorems to show the range of techniques available and uses examples to link enumerative combinatorics to other areas of study. The main section of the book introduces the key tools of the subject (generating functions and recurrence relations), which are then used to study the most important combinatorial objects, namely subsets, partitions, and permutations of a set. Later chapters deal with more specialised topics, including permanents, SDRs, group actions and the Redfield–Pólya theory of cycle indices, Möbius inversion, the Tutte polynomial, and species.

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CAMBRIDGE
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University Printing House, Cambridge CB2 8BS, United Kingdom
One Liberty Plaza, 20th Floor, New York, NY 10006, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
4843/24, 2nd Floor, Ansari Road, Daryaganj, Delhi – 110002, India
79 Anson Road, #06-04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of
education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781108417365
DOI: 10.1017/9781108277457

First published 2017

Printed in the United Kingdom by Clays, St Ives plc

A catalogue record for this publication is available from the British Library.

ISBN 978-1-108-41736-5 Hardback
ISBN 978-1-108-40495-2 Paperback

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Preface

Combinatorics is the science of pattern and arrangement. A typical problem in combinatorics asks whether it is possible to arrange a collection of objects according to certain rules. If the arrangement is possible, the next question is a counting question: how many different arrangements are there? This is the topic of the present book.

Often a small change in the detail of a problem turns an easy question into one which appears impossibly difficult. For example, consider the following three questions.

- In how many ways can the numbers $1,\ldots,n$ be placed in the cells of an $n\times n$ grid, with no restriction on how many times each is used? Since each of the n^2 cells can have its entry chosen independently from a set of n possibilities, the answer is n^{n^2}.
- In how many ways can the arrangement be made if each number must occur once in each row? Once we notice that each row must be a permutation of the numbers $1,\ldots,n$, and that the permutations can be chosen independently, we see that the answer is $(n!)^n$ (as there are $n!$ permutations of the numbers $1,\ldots,n$).
- In how many ways can the arrangement be made if each number must occur once in each row and once in each column? For this problem, there is no formula for the answer. Such an arrangement is called a *Latin square*. The number of Latin squares with n up to 11 has been found by brute-force calculation. For larger values, we don't even have good estimates: the best known upper and lower bounds differ by a factor which is exponentially large in terms of the number of cells.

Not all problems are as hard as this. In this book you will learn how to count the number of permutations which move every symbol, strings of zeros and ones containing no occurrence of a fixed substring, invertible matrices of given size over a finite field, expressions for a given positive integer as a sum of positive integers (the answers are different depending on whether we care about the order of the summands or not), trees and graphs on a given set of vertices, and many more.

Very often I will not be content with giving you one proof of a theorem. I may return to an earlier result armed with a new technique and give a totally different proof. We learn something from having several proofs of the same result. As Michael Atiyah said in an interview in the *Newsletter of the European Mathematical Society*,

> I think it is said that Gauss had ten different proofs for the law of quadratic reciprocity. Any good theorem should have several proofs, the more the better. For two reasons: usually, different proofs have different strengths and weaknesses, and they generalise in different directions — they are not just repetitions of each other.

In particular, there are two quite different styles of proof for results in enumerative combinatorics. Consider a result which asserts that two counting functions $F(n)$ and $G(n)$ are equal. We might prove this by showing that their generating functions are equal, perhaps using analytic techniques of some kind. Alternatively, we might prove the result by finding a bijection between the sets of objects counted by $F(n)$ and $G(n)$. Often, when such an identity is proved by analytic methods, the author will ask for a 'bijective proof' of the result.

As an example, if n is even, then it is fairly straightforward to prove by analytic methods that the number of permutations of $\{1,\ldots,n\}$ with all cycles even is equal to the number with all cycles odd. But finding an explicit bijection between the two sets is not straightforward, though not too difficult.

I should stress, though, that the book is not full of big theorems. Tim Gowers, in a perceptive article on 'The two cultures of mathematics', distinguishes branches of mathematics in which theorems are all-important from those where the emphasis is on techniques; enumerative combinatorics falls on the side of techniques. (In the past this has led to some disparagement of combinatorics by other mathematicians. Many people know that Henry Whitehead said 'Combinatorics is the slums of topology'. A more honest appraisal is that the techniques of combinatorics pervade all of mathematics, even the most theorem-rich parts.)

The notes which became this book were for a course on *Enumerative and Asymptotic Combinatorics* at Queen Mary, University of London, in the spring of 2003, and subsequently as *Advanced Combinatorics* at the University of St Andrews. The reference material for the subject has been greatly expanded by the publication of Richard Stanley's two-volume work on *Enumerative Combinatorics*, as well as the book on *Analytic Combinatorics* by Flajolet and Sedgewick. (References to these and many other books can be found in the bibliography at the end.) Many of these books are encyclopaedic in nature. I hope that this book will be an introduction to the subject, which will encourage you to look further and to tackle some of the weightier tomes.

What do you need to know to read this book? It will probably help to have had some exposure to basic topics in undergraduate mathematics.

- Real and complex analysis (limits, convergence, power series, Cauchy's theorem, singularities of complex functions);
- Abstract algebra (groups and rings, group actions);
- Combinatorics.

None of this is essential; in most cases you can pick up the needed material as you go along.

The heart of the book is Chapters 2–4, in which the most important tools of the subject (generating functions and recurrence relations) are introduced and used to study the most important combinatorial objects (subsets, partitions and permutations of a set). The basic object here is a *formal power series*, a single object encapsulating an infinite sequence of numbers, on which a wide variety of manipulations can be done: formal power series are introduced in Chapter 2.

Later chapters treat more specialised topics: permanents, systems of distinct representatives, and Latin squares in Chapter 5, 'q-analogues' (familiar formulae with an extra parameter arising in a wide variety of applications) in Chapter 6, group actions and the Redfield–Pólya theory of the cycle index in Chapter 7, Möbius inversion (a wide generalisation of the Inclusion–Exclusion Principle) in Chapter 8, the Tutte polynomial (a counting tool related to Inclusion–Exclusion) in Chapter 9, species (an abstract formalism including many important counting problems) in Chapter 10, and some miscellaneous topics (mostly analytic) in Chapters 11 and 12. The final chapter includes an annotated list of books for further study.

As always in a combinatorics book, the techniques described have unexpected applications, and it is worth looking through the index. Cayley's Theorem on trees, for example, appears in Chapter 10, where several different proofs are given; Young tableaux are discussed in Chapter 4, as are various counts of inverse semigroups of partial permutations.

The final chapter includes an annotated book list and a discussion of using the On-line Encyclopedia of Integer Sequences.

The first few chapters contain various interdependences. For example, binomial coefficients and the Binomial Theorem for natural number exponents appear in Chapter 2, although they are discussed in more detail in Chapter 3. Such occurrences will be flagged, but I hope that you will have met these topics in undergraduate courses or will be prepared to take them on trust when they first appear.

I am grateful to many students who have taken this course (especially Pablo Spiga, Thomas Evans, and Wilf Wilson), to colleagues who have helped teach

it (especially Thomas Müller), and to others who have provided me with examples (especially Thomas Prellberg and Dudley Stark), and to Abdullahi Umar for the material on inverse semigroups. I am also grateful to Morteza Mohammed-Noori, who used my course notes for a course of his own in Tehran, and did a very thorough proof-reading job, spotting many misprints. (Of course, I may have introduced further misprints in the rewriting!)

The book will be supported by a web page at

`http://www-circa.mcs.st-and.ac.uk/~pjc/books/counting/`

which will have a list of misprints, further material, links, and possibly solutions to some of the exercises.