Ordinary Differential Equations

Many interesting and important real life problems in the field of mathematics, physics, chemistry, biology, engineering, economics, sociology and psychology are modelled using the tools and techniques of ordinary differential equations (ODEs). This book offers detailed treatment on fundamental concepts of ordinary differential equations. Important topics including first and second order linear equations, initial value problems and qualitative theory are presented in separate chapters. The concepts of physical models and first order partial differential equations are discussed in detail. The text covers two-point boundary value problems for second order linear and nonlinear equations. Using two linearly independent solutions, a Green's function is also constructed for given boundary conditions.

The text emphasizes the use of calculus concepts in justification and analysis of equations to get solutions in explicit form. While discussing first order linear systems, tools from linear algebra are used and the importance of these tools is clearly explained in the book. Real life applications are interspersed throughout the book. The methods and tricks to solve numerous mathematical problems with sufficient derivations and explanations are provided.

The first few chapters can be used for an undergraduate course on ODE, and later chapters can be used at the graduate level. Wherever possible, the authors present the subject in a way that students at undergraduate level can easily follow advanced topics, such as qualitative analysis of linear and nonlinear systems.

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We would like to dedicate the book to our parents
who brought us to this wonderful world.

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Preface

Many interesting and important real life problems are modeled using ordinary differential equations (ODE). These include, but are not limited to, physics, chemistry, biology, engineering, economics, sociology, psychology etc. In mathematics, ODE have a deep connection with geometry, among other branches. In many of these situations, we are interested in understanding the future, given the present phenomenon. In other words, we wish to understand the time evolution or the dynamics of a given phenomenon. The subject field of ODE has developed, over the years, to answer adequately such questions. Yet, there are many important intriguing situations, where complete answers are still awaited. The present book aims at giving a good foundation for a beginner, starting at an undergraduate level, without compromising on the rigour.

We have had several occasions to teach the students at the undergraduate and graduate level in various universities and institutions across the country, including our own institutions, on many topics covered in the book. In our experience and the interactions we have had with the students, we felt that many students lack a clear notion of ODE including the simplest integral calculus problem. For other students, a course on ODE meant learning a few tricks to solve equations. In India, in particular, the books which are generally prescribed, consist of a few tricks to solve problems, making ODE one of the most uninteresting subject in the mathematical curriculum. We are of the opinion that many students at the beginning level do not have clarity about the essence of ODE, compared to other subjects in mathematics.

While we were still contemplating to write a book on ODE, to address some of the issues discussed earlier, we got an opportunity to present a video course on ODE, under the auspices of the National Programme for Technology Enhanced Learning (NPTEL), Department of Science and Technology (DST), Government of India, and our course is freely available on the NPTEL website (see www.nptel.ac.in/courses/111108081). In this video course, we have presented several topics. We have also tried to address many of the doubts that students may have at the beginning level and the misconceptions some other students may possess.

Many in the academic fraternity, who watched our video course, suggested that we write a book. Of course, writing a text book, that too about a classical subject at a beginning level, meant a much bigger task than a video course, involving choosing and presenting the material in a very systematic way. In a way, the video course may supplement the book as it gives a flavour of a classroom lecture. We hope that in this way, students in remote areas and/or places where there is lack of qualified teachers, benefit from the book and the video course, making good use of the modern technology available through the Internet. The teachers of undergraduate courses can also benefit, we hope, from this book in fine tuning their skills in ODE.

We have written the present book with the hope that it can also be used at the undergraduate level in universities everywhere, especially in the context of Indian universities, with appropriately chosen topics in Chapters 1, 2 and 3. As the students get more acquainted with basic analysis and linear algebra, the book can be introduced at the graduate level as well and even at the beginning level of a research programme.

We now briefly describe the contents of the book. The book has a total of ten chapters and one appendix.

Chapter 1 describes some important examples from real life situations in the field of physics to biology to engineering. We thought this as a very good motivation for a beginner to undertake the study of ODE; in a rigorous course on ODE, often a student does not see a good reason to study the subject. We have observed that this has been one of the major concerns faced by students at a beginning level.

As far as possible, we have kept the prerequisite to a minimum: a good course on calculus. With this in mind, we have collected, in Chapter 2, a number of important results from analysis and linear algebra that are used in the main text. Wherever possible, we have provided proofs and simple presentations. This makes the book more or less self contained, though a deeper knowledge in analysis and linear algebra will enhance the understanding of the subject.

First and second order equations are dealt with in Chapter 3. This chapter also contains the usual methods of solutions, but with sufficient mathematical explanation, so that students feel that there is indeed rigorous mathematics behind these methods. The concept behind the exact differential equation is also explained. Second order linear equations, with or without constant coefficients, are given a detailed treatment. This will make a student better equipped to study linear systems, which are treated in Chapter 5.

Chapter 4 deals with the hard theme of existence, non-existence, uniqueness etc., for a single equation and also a system of first order equations. We have tried to motivate the reader to wonder why these questions are important and how to deal with them. We have also discussed other topics such as continuous dependence on initial data, continuation of solutions and the maximal interval of existence of a solution.

Linear systems are studied in great detail in Chapter 5. We have tried to show the power of linear algebra in obtaining the phase portrait of 2×2 and general systems. We have also included a brief discussion on Floquet theory, which deals with linear systems with periodic coefficients.

In the case of a second order linear equation with variable coefficients, it is not possible in general, to obtain a solution in explicit form. This has been discussed at length in Chapter 3. Chapter 6 deals with a class of second order linear equations, whose solutions may be written explicitly, although in the form of an infinite series. This method is attributed to Frobenius.

Chapter 7 deals with the regular Sturm–Lioville theory. This theory is concerned with boundary value problems associated with linear second order equations with smooth coefficients, in a compact interval on the real, involving a parameter. We, then, show the existence of a countable number of values of the parameter and associated non-trivial solutions of the differential equation satisfying the boundary conditions. There are many similarities with the existence of eigenvalues and eigenvectors of a matrix, though we are now in an infinite dimensional situation.

The qualitative theory of nonlinear systems is the subject of Chapter 8. The contents may be suitable for a senior undergraduate course or a beginning graduate course. This chapter does demand for more prerequisites and these are described in Chapter 2. The main topics of the chapter are equilibrium points or solutions of autonomous systems and their stability analysis; existence of periodic orbits in a two-dimensional system. We have tried to make a presentation of these important notions so that it can be easily understood by any student at a senior undergraduate level. The proofs of two important theorems on the existence of periodic orbits are given in the Appendix.

Chapter 9 considers the study of two point boundary value problems for second order linear and nonlinear equations. The first dealing with linear equations fully utilises the theory developed in Chapter 3. Using two linearly independent solutions, a Green's function is constructed for given boundary conditions. This is similar to an integral calculus problem. For nonlinear equations, we no longer have the luxury of two linearly independent solutions. A result which gives a taste of delicate analysis is proved. It is also seen through some examples how phase plane analysis can help in deciding whether a given boundary value problem has a solution or not.

In Chapter 10, we have attempted to show how the methods of ODE are used to find solutions of first order partial differential equations (PDE). We essentially describe the method of characteristics for solving general first order PDE. As very few books on ODE deal with this topic, we felt like including this, as a student gets some benefit of studying PDE and (s)he can later pursue a course on PDE.

We have followed the standard notations. Vectors in Euclidean faces and matrices are in boldface.

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We wish to express our sincere appreciation to Gadadhar Misra and others at the IISc Press for suggesting to publish our book through the joint venture of IISc Press and Cambridge University Press. We also would like to thank Gadadhar Misra for all the help in this regard. We wish to acknowledge the support we received from our respective institutions and the moral support from our colleagues, during the preparation of the manuscript. We thank our academic fraternity, who have made valuable suggestions after reading through the various parts of the book. We would like to thank the students who attended our lectures at various places and contributed in a positive way. Over the years, we have had the opportunity to deliver talks in various lecture programs conducted by National Programme in Differential Equations (NPDE), India and the Indian Science Acadamies; our sincere thanks to them. We also wish to thank the anonymous referees for their constructive criticism and suggestions, which have helped us in improving the presentation. The illustrations have been drawn using the freely available software packages tikz and circuitikz. We are also thankful to the CUP team for their coordination from the beginning and their excellent production. Last but not the least, we wish to thank our family members for their patience and support during the preparation of this book.