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978-1-108-41641-2 — Ordinary Differential Equations
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Introduction and Examples: Physical Models

1.1 A Brief General Introduction

The beginning of the study of ordinary differential equations (ODE) could perhaps be attributed to Newton and Leibnitz, the inventors of differential and integral calculus. The theory began in the late 17th century with the early works of Newton, Leibnitz and Bernoulli. As was customary then, they were looking at the fundamental problems in geometry and celestial mechanics. There were also important contributions to the development of ODE, in the initial stages, by great mathematicians – Euler, Lagrange, Laplace, Fourier, Gauss, Abel, Hamilton and others. As the modern concept of function and analysis were not developed at that time, the aim was to obtain solutions of differential equations (and in turn, solutions to physical problems) in terms of elementary functions. The earlier methods in this direction are the concepts of *integrating factors* and method of separation of variables.

In the process of developing more systematic procedures, Euler, Lagrange, Laplace and others soon realized that it is hopeless to discover methods to solve differential equations. Even now, there are only a handful of sets of differential equations, that too in a simpler form, whose solutions may be written down in explicit form. It is in this scenario that the *qualitative analysis – existence, uniqueness, stability properties, asymptotic behaviour* and so on – of differential equations became very important. This qualitative analysis depends on the development of other branches of mathematics, especially analysis. Thus, a second phase in the study of differential equations started from the beginning of the 19th century based on a more rigorous approach to calculus via the

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mathematical analysis. We remark that the first existence theorem for first order differential equations is due to Cauchy in 1820. A class of differential equations known as *linear differential equations*, is much easier to handle. We will analyse linear equations and linear systems in more detail and see the extensive use of linear algebra; in particular, we will see how the nature of eigenvalues of a given matrix influences the stability of solutions.

After the invention of differential calculus, the question of the existence of antiderivative led to the following question regarding differential equation: Given a function f, does there exist a function g such that $\dot{g}(t) = f(t)$? Here, $\dot{g}(t)$ is the derivative of g with respect to t. This was the beginning of integral calculus and we refer to this problem as an *integral calculus problem*. In fact, Newton's second law of motion describing the motion of a particle having mass m states that the rate change of momentum equals the applied force. Mathematically, this is written as $\frac{d}{dt}(mv) = -F$, where v is the velocity of the particle. If $x = x(t)$ is the position of the particle at time t, then $v(t) = \dot{x}(t)$. In general, the applied force F is a function of t, x and v. If we assume F is a function of t, x, we have a second order equation for x given by $m\ddot{x} = -F(t,x)$. If F is a function of x alone, we obtain a conservative equation which we study in Chapter 8. If on the other hand, F is a function of t alone, then the second law leads to two integral calculus problems: namely, first solve for the momentum $p = mv$ by $\dot{p} = -F(t)$ and then solve for the position using $m\dot{x} = p$. This also suggests that one of the best ways to look at a differential equation is to view it as a *dynamical system*; namely, the motion of some physical object. Here t, the *independent variable* is viewed as time and x is the unknown variable which depends on the independent variable t, and is known as the *dependent variable*.

A large number of physical and biological phenomena can be modelled via differential equations. Applications arise in almost all branches of science and engineering–radiation decay, aging, tumor growth, population growth, electrical circuits, mechanical vibrations, simple pendulum, motion of artificial satellites, to mention a few.

In summary, real life phenomena together with physical and other relevant laws, observations and experiments lead to mathematical models (which could be ODE). One would like to do mathematical analysis and computations of solutions of these models to simulate the behaviour of these physical phenomena for better understanding.

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Definition 1.1.1

An ODE is an equation consisting of an independent variable t, an unknown function (dependent variable) $y = y(t)$ and its derivatives up to a certain order. Such a relation can be written as

$$f\left(t, y, \frac{dy}{dt}, \cdots, \frac{d^n y}{dt^n}\right) = 0. \tag{1.1.1}$$

Here, n is a positive integer, known as the order of the differential equation.

For example, first and second order equations, respectively, can be written as

$$f\left(t, y, \frac{dy}{dt}\right) = 0 \text{ and } f\left(t, y, \frac{dy}{dt}, \frac{d^2y}{dt^2}\right) = 0. \tag{1.1.2}$$

We will be discussing some special cases of these two classes of equations. It is possible that there will be more than one unknown function and in that case, we will have a system of differential equations. A higher order differential equation in one unknown function may be reduced into a system of first order differential equations. On the other hand, if there are more than one independent variable, we end up with partial differential equations (PDEs).

1.2 Physical and Other Models

We begin with a few mathematical models of some real life problems and present solutions to some of these problems. However, methods of obtaining such solutions will be introduced in Chapter 3, and so are the terminologies like linear and nonlinear equations.

1.2.1 Population growth model

We begin with a linear model. If $y = y(t)$, represents the population size of a given species at time t, then the rate of change of population $\frac{dy}{dt}$ is proportional to $y(t)$ if there is no other species to influence it and there is no net migration. Thus, we have a simple linear model [Bra78]

$$\frac{dy}{dt} = ry(t), \tag{1.2.1}$$

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where r denotes the difference between birth rate and death rate. If $y(t_0) = y_0$ is the population at time t_0, our problem is to find the population for all $t > t_0$. This leads to the so-called *initial value problem (IVP)* which will be discussed in Chapter 3. Assuming that r is a constant, the solution is given by

$$y(t) = y_0 e^{r(t-t_0)} \tag{1.2.2}$$

Note that, if $r > 0$, then as $t \to \infty$, the population $y(t) \to \infty$. Indeed, this linear model is found to be accurate when the population is small and for small time. But it cannot be a good model as no population, in reality, can grow indefinitely. As and when the population becomes large, there will be competition among the population entities for the limited resources like food, space etc.

This suggests that we look for a more realistic model which is given by the following logistic nonlinear model. The statistical average of the number of encounters of two members per unit time is proportional to y^2. Thus, a better model would be

$$\frac{dy}{dt} = ay - by^2, \; y(t_0) = y_0. \tag{1.2.3}$$

Here a, b are positive constants. The negative sign in the quadratic term represents the competition and reduces the growth rate. This is known as the *logistic law of population growth*. It was introduced by the Dutch mathematical biologist Verhulst in 1837. It is also known as the Malthus law.

Practically, b is small compared to a. Thus, if y is not too large, then by^2 will be negligible compared to ay and the model behaves similar to the linear model. However, when y becomes large, the term by^2 will have a considerable influence on the growth of y, as can be seen from the following discussion.

The solution of (1.2.3) is given by[1]

$$\frac{1}{a} \log \frac{|y|}{|y_0|} \left| \frac{a - by_0}{a - by} \right| = t - t_0, \; t > t_0. \tag{1.2.4}$$

Note that $y \equiv 0$ and $y \equiv \frac{a}{b}$ are solutions to the nonlinear differential equation in (1.2.3) with the initial condition $y(t_0) = 0$ and $y(t_0) = \frac{a}{b}$,

[1]The reader, after getting familiarised with the methods of solutions in Chapter 3, should work out the details for this and the other examples in this chapter.

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respectively. Hence, if the initial population y_0 satisfies $0 < y_0 < \frac{a}{b}$, then the solution will remain in the same interval for all time. This follows from the existence and uniqueness theory, which will be developed in Chapter 4. A simplification of (1.2.4) gives

$$y(t) = \frac{ay_0}{by_0 + (a - by_0)e^{-a(t-t_0)}}. \tag{1.2.5}$$

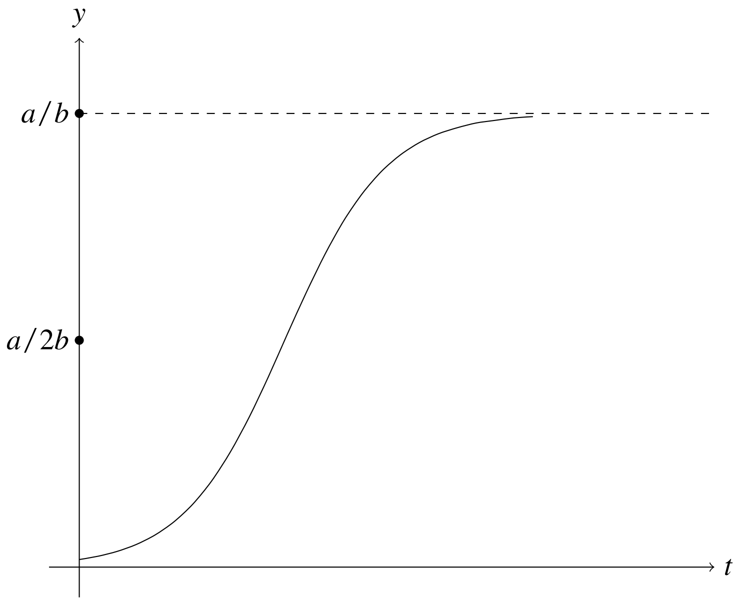

Fig. 1.1 Logistic map

In case $0 < y_0 < \dfrac{a}{b}$, the curve $y(t)$ is depicted as in Fig. 1.1. This curve is called the *logistic* curve; it is also called an *S-shaped curve*, because of its shape. Note that $\dfrac{a}{b}$ is the limiting population, also known as capacity of the ecological environment. In this case, the rate of population $\dfrac{dy}{dt}$ is positive and hence, y is an increasing function. Since $\dfrac{d^2y}{dt^2} = (a - 2by)\dfrac{dy}{dt}$, we immediately see that it is positive if the population is between 0 and half the limiting population, namely, $\dfrac{a}{2b}$, whereas, it is negative when the

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population crosses the half way mark $\frac{a}{2b}$. This indicates that if the initial population is less than half the limiting population, then there is an accelerated growth $\left(\frac{dy}{dt} > 0, \frac{d^2y}{dt^2} > 0\right)$, but after reaching half the population, the population still grows $\left(\frac{dy}{dt} > 0\right)$, but it has now a decelerated growth $\left(\frac{d^2y}{dt^2} < 0\right)$.

When we analyse the case where the initial population is bigger than the limiting population, we observe that $\frac{dy}{dt} < 0$ and $\frac{d^2y}{dt^2} < 0$. Thus, the population *decreases with a decelerated growth* to the limiting population.

Remark 1.2.1

The estimation of the vital coefficients a and b in a particular population model is indeed an important issue which has to be updated in a period of time as they are influenced by other parameters like pollution, sociological trends, etc. In a more realistic model, one needs to consider more than one species, their interactions, unforeseen issues like epidemics, natural disasters, etc., which may lead to more complicated equations.

1.2.2 An atomic waste disposal problem

The dumping of tightly sealed drums containing highly concentrated radioactive waste in the sea below a certain depth (say 300 feet) from the surface is a very sensitive issue as it could be environmentally hazardous. The drums could break due to the impact of their velocity exceeding a certain limit, say 40 ft/sec. Our problem is to compute the velocity by using Newton's second law of motion and assess the level of safety involved in the process. Let $y(t)$ denote the position, at time t, of the object, the drum, (considered as a particle) measured from the sea surface (indicating $y = 0$) as a positive quantity. The total force acting on the object is given by

$$F = W - B - D,$$

where the weight $W = mg$ is the force due to gravity, B is the buoyancy force of water acting against the forward movement and $D = cV$ is the drag

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exerted by water (it is a kind of resistance), where $V = \frac{dy}{dt}$, the velocity of the object and $c > 0$ is a constant of proportionality. Thus, we have the differential equation

$$\frac{d^2y}{dt^2} = \frac{1}{m}F = \frac{1}{m}(W - B - cV) = \frac{g}{W}(W - B - cV),\ y(0) = 0. \quad (1.2.6)$$

Equivalently,

$$\frac{dV}{dt} + \frac{cg}{W}V = \frac{g}{W}(W - B),\ V(0) = 0. \quad (1.2.7)$$

Equation (1.2.7) can be solved to get

$$V(t) = \frac{W - B}{c}\left(1 - e^{-\frac{cg}{W}t}\right). \quad (1.2.8)$$

Thus, $V(t)$ is increasing and tends to $\frac{W-B}{c}$ as $t \to \infty$ and the value (practically) of $\frac{W - B}{c} \approx 700$.

The limiting value 700 ft/sec of velocity is far above the permitted critical value. Thus, it remains to ensure that $V(t)$ does not reach 40 ft/sec by the time it reaches the sea bed. But it is not possible to compute t at which time the drum hits the sea bed and one needs to do further analysis.

Analysis: The idea is to view the velocity $V(t)$ not as a function of time, but as a function of position y. Let $v(y)$ be the velocity at height y measured from the surface of the sea downwards. Then, clearly, $V(t) = v(y(t))$ so that $\frac{dV}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}$. Hence, (1.2.7) becomes

$$\begin{cases} \dfrac{v}{W - B - cv}\dfrac{dv}{dy} = \dfrac{g}{W}, \\ v(0) = 0. \end{cases} \quad (1.2.9)$$

This is a first order non-homogeneous nonlinear equation for the velocity v. Indeed, the equation is more difficult, but it is in a variable separable form and can be integrated easily. We can solve this equation to obtain the solution in the form

$$\frac{gy}{W} = -\frac{v}{c} - \frac{W - B}{c^2}\log\frac{W - B - cv}{W - B}. \quad (1.2.10)$$

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Of course, v cannot be explicitly expressed in terms of y as it is a nonlinear equation. However, it is possible to obtain accurate estimates for the velocity $v(y)$ at height y and it is estimated that $v(300) \approx 45$ ft/sec and hence, the drum could break at a depth of 300 feet.

Tail to the Tale: This problem was initiated when environmentalists and scientists questioned the practice of dumping waste materials by the Atomic Energy Commission of USA. After the study, the dumping of atomic waste was forbidden, in regions of sea not having sufficient depths.

1.2.3 Mechanical vibration model

The fundamental mechanical model, namely *spring-mass-dashpot system (SMD)* has applications in shock absorbers in automobiles, heavy guns, etc. An object of mass m is attached to an elastic spring of length l which is suspended from a rigid horizontal body. This is a spring–mass system. Elastic spring has the property that when it is stretched or compressed by a small length Δl, it will exert a force of magnitude proportional to Δl, say $k\Delta l$ in the opposite direction of stretching or compressing. The positive constant k is called *spring constant* which is a measure of stiffness of the spring. We then obtain an SMD system when this spring–mass is immersed in a medium like oil which will also resist the motion of the spring–mass. In a simple situation, we may assume that the force exerted by the medium on the spring–mass is proportional to the velocity of the mass and in the opposite direction of the movement of mass. It is also similar to a seismic instrument used to obtain a seismograph to detect the motion of the earth's surface.

Let $y(t)$ denote the position of mass at time t, $y = 0$ being the position of the mass at equilibrium and let us take the downward direction as positive. There are four forces acting on the system, that is, $F = W + R + D + F_0$, where $W = mg$, the force due to gravity; $R = -k(\Delta l + y)$, the restoring force; D, the damping or drag force and F_0, the external applied force, if any. Drag force is the kind of resistance force which the medium exerts on the mass and hence, it will be negative. It is usually proportional to the velocity, that is, $D = -c\dfrac{dy}{dt}$. At equilibrium, the spring has been stretched a length Δl and so $k\Delta l = mg$. Applying Newton's second law, we get

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$$m\frac{d^2y}{dt^2} = -ky - c\frac{dy}{dt} + F_0(t). \tag{1.2.11}$$

That is,

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F_0(t),\ m,c,k \geq 0. \tag{1.2.12}$$

This is a second order non-homogeneous linear equation with constant coefficients and we study such equations in detail in Chapter 3. Such a system also arises in electrical circuits, which we discuss next.

1.2.4 Electrical circuit

A basic LCR electrical circuit is shown in Fig. 1.2, and is described as follows:

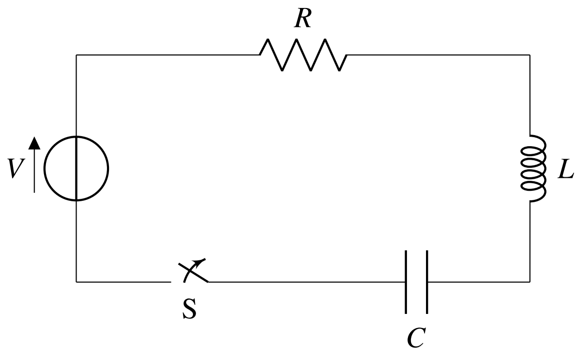

Fig. 1.2 A basic LCR circuit

By Kirchoff's second law, the impressed voltage in a closed circuit equals the sum of the voltage drops in the rest of the circuit. Let $E(t)$ be the source of electro motive force (emf), say a battery, $I = \frac{dQ}{dt}$ be the current flow, $Q(t)$ the charge on the capacitor at time t. Then, the voltage drops across inductance (L), resistance (R) and capacitance (C), respectively, are given by $L\frac{dI}{dt} = L\frac{d^2Q}{dt^2}$, $RI = R\frac{dQ}{dt} + \frac{Q}{c}$. Thus, we obtain a similar equation for Q as in (1.2.12):

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{c} = E(t). \tag{1.2.13}$$

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More often, the current $I(t)$ is the physical quantity of interest; by differentiating (1.2.13) with respect to t, the equation satisfied by I is

$$L\frac{d^2I}{dt^2}+R\frac{dI}{dt}+\frac{1}{c}I=\frac{dE}{dt}(t). \tag{1.2.14}$$

Mathematically, the equation is exactly same as the equation obtained in the spring–mass–dashpot system. We can also see the similarity between various quantities: inductance corresponding to mass, resistance corresponding to damping constant and so on.

1.2.5 Satellite problem

Consider an artificial satellite of mass m orbiting the earth. We assume that the satellite has thrusting capacity with radial thrust u_1 and a thrust u_2 which is applied in a direction perpendicular to the radial direction. The thrusters u_1 and u_2 are considered as the external force F or control inputs applied to the satellite.

The satellite can be considered as a particle P moving around the earth in the equatorial plane. If $(x,y)$is the rectangular coordinate of the particle P of mass m, then by Newton's law, the equations of motion along the rectangular coordinate axes are given by

$$m\ddot{x}=F_x, \quad m\ddot{y}=F_y \tag{1.2.15}$$

where, F_x and F_y denote the components of the force F in the directions of the axes (see Fig. 1.3). It will be convenient to represent the motion in polar coordinates (r,θ), where,

$$x=r\cos\theta,\ \ y=r\sin\theta$$

We will resolve the velocity, acceleration and force of the particle into components along the radial direction and the direction perpendicular to it. Denote by u, v; a_1, a_2 and F_r, F_θ the components of velocity, acceleration and force, respectively in the new coordinate system. The resultant of u and v is also equal to the resultant of the components of $\dot{x}$ and $\dot{y}$. Therefore, by resolving parallel to the x-axis, we get

$$\dot{x} \ \ = \ \ u\cos\theta - v\sin\theta \tag{1.2.16}$$

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