Finite Elements

Finite element method (FEM) is a numerical technique for approximation of the solution of partial differential equations. This book on FEM and related simulation tools starts with a brief introduction to Sobolev spaces and elliptic scalar problems and continues with explanation of finite element spaces and estimates for the interpolation error. Construction of finite elements on simplices, quadrilaterals, hexahedral is discussed by the authors in self-contained manner. The last chapter focuses on object-oriented finite element algorithms and efficient implementation techniques. Readers can also find the concepts of scalar parabolic problems and high dimensional parabolic problems in different chapters. Besides, the book includes some recent advances in FEM, including nonconforming finite elements for boundary value problems of higher order and approaches for solving differential equations in high-dimensional domains. There are plenty of solved examples and mathematical theorems interspersed throughout the text for better understanding.

Sashikumaar Ganesan is Assistant Professor at the Indian Institute of Science (IISc). He heads the research group on Computational Mathematics at the Department of Computational and Data Sciences at IISc. His areas of interest include numerical analysis, finite elements in fluid dynamics, moving boundary value problems, numerical methods for turbulent flows, variational multiscale methods, population balance modelling, scalable algorithms and high performance computing.

Lutz Tobiska is Professor at the Institute for Analysis and Computational Mathematics, Otto-von-Guericke University Magdeburg, Germany. His research interests include finite elements in fluid dynamics, parallel algorithms, multigrid methods and adaptive methods for convection diffusion equations.

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FINITE ELEMENTS

Theory and Algorithms

Sashikumaar Ganesan

Lutz Tobiska

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Preface

The purpose of this book is to present—in a coherent and lucid way—the mathematical theory and algorithms of the finite element method, which is the most widely-used method for the solution of partial differential equations in the field of Computational Science. We believe that the full potential of the finite element method can be realised only when the theoretical background and the implemented algorithms are considered as a unit.

The selection of the basic mathematical theory of finite elements in this book is based on lectures given in the "Finite Element I and II" courses offered for several years at Otto-von-Guericke University, Magdeburg and in the "Finite Element I" course offered at the Indian Institute of Science, Bangalore. Furthermore, the finite element algorithms presented are based on the knowledge and experience gained through the development of our in-house finite element package for more than 10 years. The theory and algorithms of finite elements that we describe here are self-contained; our aim is that beginners will find our book to be both readable and useful.

We start in Chapter 1 with a brief introduction to Sobolev spaces and the necessary basics of functional analysis. This will help those readers who are unfamiliar with functional analysis. The goal of Chapter 2 is to explain the finite element method to beginners in the simplest possible way. The concepts of weak solutions, variational formulation of second-order elliptic boundary value problems, incorporation of different boundary conditions in a variational form, and the standard Galerkin approach are all introduced in this chapter. Moreover, existence and uniqueness theory (the Lax-Milgram Theorem) and an abstract error analysis (quasi-optimality of the method) are presented here.

The next two Chapters give the basic theory of the finite element method. In Chapter 3, the construction of finite elements on simplices, quadrilaterals, and hexahedrals is discussed in detail. Furthermore, linear, bilinear and isoparametric transformations are explained, and mapped finite elements are considered. Chapter 4 deals with the interpolation theory of affine equivalent finite elements in Sobolev spaces. We also discuss the interpolation of functions that are less smooth; in particular, Scott-Zhang interpolation is described in this Chapter.

Finite elements for more advanced scalar problems are presented in Chapters 5 and 6. As an example of a finite element method for elliptic higher-order problems, conforming and nonconforming finite element methods for the biharmonic equation are studied in Chapter 5. Next, the finite element method for scalar parabolic problems is presented in Chapter 6. This includes a discussion of the standard θ-methods and discontinuous Galerkin methods for temporal discretisation. In addition, splitting schemes in the context of finite element methods for high-dimensional scalar parabolic problems are examined.

Finite element methods for systems of equations in solid mechanics and fluid mechanics are presented in Chapters 7 and 8. In particular, finite element methods for systems of equations in linear elasticity and for Mindlin-Reissner plate problems are discussed in Chapter 7. Chapter 8 deals in detail with finite element methods for the Stokes and Navier-Stokes equations. Starting from the derivation of conservation of mass and momentum equations, we discuss the implementation of different boundary conditions, a mixed variational formulation, the necessary condition for the stability of the finite element scheme, and finite elements that satisfy that stability condition. Furthermore, the derivation of a priori error estimates for Stokes problems, conforming and nonconforming finite element discretizations, and an array of inf-sup finite element pairs in two and three dimensions are presented. As an alternative to mixed finite element methods, stabilized finite elements with equal-order interpolations that circumvent the inf-sup stability are discussed. Hints on the choice of pressure spaces and on linearization strategies for the Navier-Stokes problem are given.

Finally, in Chapter 9, finite element algorithms and implementations based on object-oriented concepts are provided. All finite element algorithms and computing tools that are needed in an efficient and robust object-oriented finite element package are presented in detail.

We believe that the theoretical material of Chapters 1 to 4 and the implementation matters of Chapter 9 should be presented in any course on finite elements. The selection of the other Chapters is a matter of taste and depends on which application is in the foreground. We have made an effort to write each chapter between Chapters 5 and 8 as a self-contained unit, so—for example—a reader who is mainly interested in partial differential equations of higher order can concentrate on our discussion of the biharmonic equation.

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