CHAPTER 1

INTRODUCTION TO SOBOLEV SPACES

In this chapter we recall some basics on functional analysis and provide a brief introduction to Sobolev spaces. For a more detailed and comprehensive study, we refer to Adams (1975).

1.1. Banach and Hilbert spaces

Definition 1.1. Let $X$ be a real linear space. A mapping $\| \cdot \| : X \rightarrow \mathbb{R}$ is called a norm on $X$ if

$\begin{align*}
&\text{1.} \quad \|x\| = 0 \iff x = 0 \text{ for all } x \in X, \\
&\text{2.} \quad \|\lambda x\| = |\lambda| \|x\| \text{ for all } x \in X, \lambda \in \mathbb{R}, \\
&\text{3.} \quad \|x + y\| \leq \|x\| + \|y\| \text{ for all } x, y \in X.
\end{align*}$

The pair $(X, \| \cdot \|)$ is called a normed space.

Setting $y = -x$ in the inequality and using the other two properties we get $\|x\| \geq 0$ for all $x \in X$.

Definition 1.2. A sequence $(x_n)_{n \in \mathbb{N}}$ is called Cauchy sequence, if for all $\varepsilon > 0$ there exists an index $n_0(\varepsilon)$ such that for all $m, n > n_0(\varepsilon)$ it holds $\|x_m - x_n\| < \varepsilon$.

Definition 1.3. A sequence $(x_n)_{n \in \mathbb{N}}$ converges to $x \in X$ if for all $\varepsilon > 0$ there is an index $n_0(\varepsilon)$ such that for all $n > n_0(\varepsilon)$ it holds $\|x_n - x\| < \varepsilon$.

Definition 1.4. A normed space $(X, \| \cdot \|)$ is called complete if every Cauchy sequence in $X$ converges in $X$.

Definition 1.5. A complete normed space is called Banach space.
Example 1.1. Let $\Omega \subset \mathbb{R}^d$, $d$ is the dimension, be an open and bounded domain. We denote by $L^p(\Omega)$, $1 \leq p < \infty$, the set of all measurable functions $f : \Omega \to \mathbb{R}$ for which

$$\int_\Omega |f(x)|^p \, dx < \infty.$$ 

Similarly, the set of all measurable functions $f : \Omega \to \mathbb{R}$ satisfying

$$\text{ess sup}\{|f(x)| : x \in \Omega\} < \infty$$

will be denoted by $L^\infty(\Omega)$. Then, $L^p(\Omega)$, $p \in [1, \infty]$ is a real linear space. Setting for $1 \leq p < \infty$

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f(x)|^p \, dx\right)^{1/p}$$

and

$$\|f\|_{L^\infty(\Omega)} := \text{ess sup}\{|f(x)| : x \in \Omega\},$$

respectively, $L^p(\Omega)$ becomes a normed space for any $p \in [1, \infty]$ if we identify functions which are equal up to a set of measure zero. This identification is necessary because we only have for a set $M \subset \Omega$ of measure zero

$$\int_\Omega |f(x)|^p \, dx = 0 \quad \Rightarrow \quad f(x) = 0 \quad \text{for all} \quad x \in \Omega \setminus M.$$ 

Strictly speaking, the elements in $L^p(\Omega)$, $p \in [1, \infty]$ are classes of equivalent functions (equal up to a set of measure zero). Then, the first condition in Definition 1.1 becomes true. The second condition follows directly from the definition of $\| \cdot \|_{L^p(\Omega)}$ and the third condition is just the inequality stated in Lemma 1.1.

Lemma 1.1 (Minkowski’s inequality). For $f, g \in L^p(\Omega)$, $p \in [1, \infty]$, we have

$$\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$ 

Theorem 1.1. $L^p(\Omega)$, $p \in [1, \infty]$, is a Banach space.

Lemma 1.2 (Hölder’s inequality). For $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ with $p, q \in [1, \infty]$, $1/p + 1/q = 1$, we have $fg \in L^1(\Omega)$ and

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$ 

Definition 1.6. Two norms $\| \cdot \|_1$ and $\| \cdot \|_2$ on a normed space $X$ are called equivalent if there exist positive constants $C_1, C_2$ such that

$$C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1 \quad \text{for all} \quad x \in X.$$ 

Note that topological properties do not change when switching to an equivalent norm, for example, a sequence is convergent in $(X, \| \cdot \|_1)$ iff it is convergent in $(X, \| \cdot \|_2)$.

Definition 1.7. Let $(X, \| \cdot \|_X)$ be a normed space. A mapping $g : X \to \mathbb{R}$ is called linear if

$$g(\alpha x + \beta y) = \alpha g(x) + \beta g(y) \quad \text{for all} \quad \alpha, \beta \in \mathbb{R}, \quad x, y \in X.$$ 

A linear mapping $g : X \to \mathbb{R}$ is continuous if there is a constant $C$ such that

$$|g(x)| \leq C\|x\|_X \quad \text{for all} \quad x \in X.$$
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**Definition 1.8.** We define the sum of two continuous linear functionals, $g_1$ and $g_2$, and the multiplication of a continuous linear functional $g$ with a real number $a$ by

$$(g_1 + g_2)(x) := g_1(x) + g_2(x) \quad \text{and} \quad (ag)(x) := ag(x), \quad a \in \mathbb{R}, \; x \in X.$$ 

Then, it can be shown that the set of all continuous linear functionals create a real linear space, the dual space $X^\ast$. In the following, for the elements $g \in X^\ast$ we use the notation $(g, x) := g(x)$.

**Lemma 1.3.** The set $X^\ast$ of continuous linear functionals $x \mapsto (g, x)$ on $X$ is a Banach space with respect to the norm

$$
\|g\|_{X^\ast} := \sup_{0 \neq x \in X} \frac{|(g, x)|}{\|x\|_X}.
$$

**Proof.** Using the definition of the dual norm $\| \cdot \|_{X^\ast}$, we see

$$
\|g\|_{X^\ast} = 0 \iff (g, x) = 0 \text{ for all } x \in X \iff g = 0.
$$

Further, we have

$$
\|\lambda g\|_{X^\ast} = \sup_{0 \neq x \in X} \frac{|\lambda (g, x)|}{\|x\|_X} = \sup_{0 \neq x \in X} \frac{\|\lambda g\|_X}{\|x\|_X} = |\lambda| \sup_{0 \neq x \in X} \frac{|(g, x)|}{\|x\|_X} = |\lambda| \|g\|_{X^\ast}
$$

and for the sum of two functionals it holds

$$
\|g_1 + g_2\|_{X^\ast} = \sup_{0 \neq x \in X} \frac{|(g_1 + g_2, x)|}{\|x\|_X} = \sup_{0 \neq x \in X} \frac{|(g_1, x) + (g_2, x)|}{\|x\|_X} \leq \sup_{0 \neq x \in X} \frac{|(g_1, x)|}{\|x\|_X} + \sup_{0 \neq x \in X} \frac{|(g_2, x)|}{\|x\|_X} = \|g_1\|_{X^\ast} + \|g_2\|_{X^\ast}.
$$

It remains to show that $(X^\ast, \| \cdot \|_{X^\ast})$ is complete. Let $(g_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $X^\ast$, that is, for all $\varepsilon > 0$ there is an index $n_0(\varepsilon)$ such that for all $m, n > n_0(\varepsilon)$ it holds $\|g_m - g_n\|_{X^\ast} < \varepsilon$.

Then, for a fixed $x \in X$, the sequence of real numbers $(\langle g_n, x \rangle)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ due to

$$
\langle g_m, x \rangle - \langle g_n, x \rangle = \langle g_m - g_n, x \rangle \leq \|g_m - g_n\|_{X^\ast} \|x\|_X < \varepsilon \|x\|_X
$$

for all $m, n > n_0(\varepsilon)$. Any Cauchy sequence of real numbers is convergent in $\mathbb{R}$, that means $\lim_{n \to \infty} \langle g_n, x \rangle = g(x)$ for all $x \in X$. The linearity of $g$ follows from

$$
g(ax + \beta y) = \lim_{n \to \infty} \langle g_n, ax + \beta y \rangle = \lim_{n \to \infty} (a \langle g_n, x \rangle + \beta \langle g_n, y \rangle) = a \lim_{n \to \infty} \langle g_n, x \rangle + \beta \lim_{n \to \infty} \langle g_n, y \rangle = ag(x) + \beta g(y).
$$

Concerning the continuity of $g$ we recall that Cauchy sequences are bounded, thus

$$
\|g_n\|_{X^\ast} \leq \|g\|_{X^\ast} \|x\|_X \leq C \|x\|_X
$$

for all $x \in X$.

The limit $n \to \infty$ shows that the linear mapping $g : X \to \mathbb{R}$ is continuous. ■
Definition 1.9. Let $V$ be a linear space. A mapping $(\cdot, \cdot) : V \times V \to \mathbb{R}$ is called an inner product on $V$ if

- $(u, v) = (v, u)$ for all $u, v \in V$,
- $(\alpha u + \beta v, w) = \alpha (u, w) + \beta (v, w)$ for all $u, v, w \in V, \alpha, \beta \in \mathbb{R}$,
- $(u, u) > 0$ for $u \neq 0$.

A norm on $V$ is induced by setting $\|u\| := \sqrt{(u, u)}$. If $(V, \| \cdot \|)$ is complete we call $V$ a Hilbert space.

Lemma 1.4 (Schwarz inequality). Let $V$ be a Hilbert space. Then,

$$|(u, v)| \leq \|u\| \|v\| \quad \text{for all } u, v \in V.$$  

Example 1.2. $V = L^2(\Omega)$ equipped with the inner product

$$(f, g) := \int_\Omega f(x)g(x) \, dx$$

is a Hilbert space.

Theorem 1.2 (Riesz representation theorem). Let $V$ be a Hilbert space. There is a unique linear mapping $R : V^* \to V$ from the dual space $V^*$ into the Hilbert space $V$ such that for all $g \in V^*$ and $v \in V$

$$(Rg, v) = (g, v) \quad \text{and} \quad \|Rg\|_V = \|g\|_{V^*}.$$  

Fixed point theorems are often used to establish the unique solvability of an abstract operator equation. Let us consider the following problem in a Banach space $V$ with an operator $P : V \to V$:

Find $u \in V$ such that $u = Pu$.  

(1.1) Solutions $u \in V$ of Problem (1.1) are called fixed points of the mapping $P : V \to V$.

Definition 1.10. The mapping $P : V \to V$ is called contractive or to be a contraction if there is a positive constant $\rho < 1$ such that

$$\|Pv_1 - Pv_2\|_V \leq \rho \|v_1 - v_2\|_V \quad \text{for all } v_1, v_2 \in V.$$  

Theorem 1.3 (Banach’s fixed point theorem). Let $V$ be a Banach space and $P : V \to V$ be a contraction. Then, we have the following statements:

1. There is a unique fixed point $u^* \in V$ solving Problem (1.1).
2. The sequence $u_n = Pu_{n-1}$, $n \geq 1$, converges to the fixed point $u^* \in V$ for any initial guess $u_0 \in V$.
3. It holds the error estimate

$$\|u_n - u^*\|_V \leq \frac{\rho^n}{1 - \rho} \|Pu_0 - u_0\|_V \quad \text{for all } n \in \mathbb{N}.$$  

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Proof. We start by estimating the distance of \( u_i \in V \) to \( u_{i-1} \in V \)

\[
\| u_i - u_{i-1} \|_V = \| Pu_{i-1} - Pu_{i-2} \|_V \leq \rho \| u_{i-1} - u_{i-2} \|_V \leq \rho^2 \| u_{i-2} - u_{i-3} \|_V \\
\vdots \quad \leq \quad \vdots \\
\| u_i - u_{i-1} \|_V \leq \rho^{i-1} \| Pu_0 - u_0 \|_V \quad \text{for all } i \in \mathbb{N}.
\]

From that we get for \( m > n \geq 1 \)

\[
\| u_m - u_n \|_V = \left\| \sum_{i=n+1}^{m} (u_i - u_{i-1}) \right\|_V \leq \sum_{i=n+1}^{m} \| u_i - u_{i-1} \|_V \\
\leq \sum_{i=n+1}^{m} \rho^{i-1} \| Pu_0 - u_0 \|_V \leq \rho^n \frac{1 - \rho^{m-n}}{1 - \rho} \| Pu_0 - u_0 \|_V \\
\leq \frac{\rho^n}{1 - \rho} \| Pu_0 - u_0 \|_V \to 0 \quad \text{for } n \to \infty.
\]

Therefore, for any \( \varepsilon > 0 \), there exists an index bound \( n_0(\varepsilon) \) such that for all \( m > n \geq n_0(\varepsilon) \)

\[
\| u_m - u_n \|_V \leq \frac{\rho^n}{1 - \rho} \| Pu_0 - u_0 \|_V < \varepsilon,
\]

consequently, \( (u_n) \) is a Cauchy sequence and converges to some \( u^* \in V \). We show that \( u^* \) is a fixed point of \( P \). Indeed,

\[
\| u^* - Pu^* \|_V \leq \| u^* - u_n \|_V + \| Pu_{n-1} - Pu^* \|_V \\
\leq \| u^* - u_n \|_V + \rho \| u_{n-1} - u^* \|_V \to 0 \quad \text{for } n \to \infty.
\]

Next we prove that there is at most one fixed point. Assume that there are two fixed points \( u_1^* \neq u_2^* \). Then, we conclude

\[
\| u_1^* - u_2^* \|_V = \| Pu_1^* - Pu_2^* \|_V \leq \rho \| u_1^* - u_2^* \|_V \quad \Rightarrow \quad (1 - \rho) \| u_1^* - u_2^* \|_V \leq 0.
\]

Since \( \rho < 1 \) we get \( u_1^* = u_2^* \). Finally, we recall the estimate for \( m > n \geq 1 \)

\[
\| u_m - u^* \|_V \leq \| u_m - u_n \|_V + \| u_m - u^* \|_V \leq \rho^n \frac{1 - \rho}{1 - \rho} \| Pu_0 - u_0 \|_V + \| u_m - u^* \|_V
\]

from which the last statement follows by \( m \to \infty \).

\[\square\]

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First we introduce a generalization of the integration by parts formula known for smooth functions \( u, v : [a,b] \to \mathbb{R} \):

\[
\int_a^b u' v \, dx = uv|_a^b - \int_a^b u v' \, dx.
\]
We start with a characterization of geometric properties of a domain $\Omega \subset \mathbb{R}^d$.

**Definition 1.11.** The domain $\Omega \subset \mathbb{R}^d$ is said to have a Lipschitz continuous boundary, if for every point of the boundary $\Gamma = \partial \Omega$ there exists a local coordinate system in which the boundary corresponds to some hypersurface with the domain $\Omega$ lying on one side of that surface and $\partial \Omega$ can locally be represented by the graph of a Lipschitz continuous mapping.

**Theorem 1.4** (Gauss theorem). Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz continuous boundary $\Gamma$ and outer normal $n$. Then, for $w \in C^1(\Omega)$ we have

$$\int_\Omega \text{div} w \, dx = \int_\Omega \sum_{i=1}^d \frac{\partial w_i}{\partial x_i} \, dx - \int_\Gamma w \cdot n \, d\gamma.$$  

Let $j \in \{1, 2, \ldots, d\}$ be a fixed index. The formula of integration by parts follows by setting $w_i = uv$ for $i = j$ and $w_i = 0$ for $i \neq j$, and using the product rule.

**Lemma 1.5** (Integration by parts). For $u, v \in C^1(\Omega)$ we have

$$\int_\Omega \frac{\partial u}{\partial x_j} v \, dx = \int_\Gamma u v n_j \, d\gamma - \int_\Omega u \frac{\partial v}{\partial x_j} \, dx, \quad j = 1, \ldots, d.$$  

Now we have the tools to generalize the concept of differentiability.

**Definition 1.12.** The support of a function $f : \Omega \to \mathbb{R}$ is defined by

$$\text{supp} f = \{x \in \Omega : f(x) \neq 0\}.$$

The set of all functions that are differentiable of any order with compact support in $\Omega$ will be denoted by $C^\infty_0(\Omega)$.

For $\Omega = (0, 1)$, the function $x \mapsto x(1-x)$ does not belong to $C^\infty_0(\Omega)$ because its support is $[0, 1] \not\subseteq (0, 1)$. Indeed, functions in $C^\infty_0(\Omega)$ are zero in the neighbourhood of the boundary $\partial \Omega$. As a consequence, a function $f \in C^\infty_0(\Omega)$ can be extended to a smooth function defined on $\mathbb{R}^d$ by setting $f = 0$ outside of $\Omega$.

**Definition 1.13.** We define

$$L^1_{\text{loc}}(\Omega) := \{f : \Omega \to \mathbb{R} : f \in L^1(A) \text{ for all compact } A \subset \Omega\}.$$  

For bounded domains $\Omega$, we have $L^2(\Omega) \subset L^1(\Omega) \subset L^1_{\text{loc}}(\Omega)$.

**Example 1.3.** Consider $\Omega = (0, 1)$. The mapping $x \mapsto 1/x$ belongs to $L^1_{\text{loc}}(\Omega)$ but not to $L^1(\Omega)$. The mapping $x \mapsto 1/\sqrt{x}$ belongs to $L^1(\Omega)$ but not to $L^2(\Omega)$.

**Definition 1.14.** Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a multi-index, that is, $\alpha_i$ are non-negative integers and $|\alpha| = \alpha_1 + \cdots + \alpha_d$. A function, $f \in L^1_{\text{loc}}(\Omega)$ has a weak derivative $\nu = D^\alpha f \in L^1_{\text{loc}}(\Omega)$ if for any $\phi \in C^\infty_0(\Omega)$

$$\int_\Omega \nu \phi \, dx = (-1)^{|\alpha|} \int_\Omega f \, D^\alpha \phi \, dx.$$
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The notion of a weak derivative generalizes the classical partial derivatives. Indeed, a partial derivative in the classical sense satisfies the above condition by partial integration.

**Example 1.4.** Let $f : (-1, +1) \to \mathbb{R}$ with $f(x) = 2 - \sqrt{|x|}$. Then, the generalized derivative $Df \in L^1_{\text{loc}}(-1, +1)$ is given by

$$Df(x) = -\frac{\text{sgn } x}{2\sqrt{|x|}} \quad x \in (-1,0) \cup (0,1).$$

**Proof.** Applying integration by parts separately in the two subintervals, we get for $\varphi \in C^\infty_0 (-1, +1)$

$$\int_{-1}^{+1} \frac{\text{sgn } x}{2\sqrt{|x|}} \varphi \, dx = \int_{-1}^{0} \frac{1}{2\sqrt{|x|}} \varphi \, dx - \int_{0}^{+1} \frac{1}{2\sqrt{|x|}} \varphi \, dx$$

$$= \left(2 - \sqrt{|x|}\right) \varphi \bigg|_{x=-1}^{0} - \int_{-1}^{0} \left(2 - \sqrt{|x|}\right) \varphi' \, dx$$

$$+ \left(2 - \sqrt{|x|}\right) \varphi \bigg|_{x=0}^{+1} - \int_{0}^{+1} \left(2 - \sqrt{|x|}\right) \varphi' \, dx$$

$$= -\int_{-1}^{+1} \left(2 - \sqrt{|x|}\right) \varphi' \, dx,$$

where we used the continuity of $\varphi$ at $x = 0$ and $\varphi(\pm 1) = 0$. 

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**Definition 1.15.** We denote by $W^{m,p}(\Omega)$, $1 \leq p \leq \infty$, $m \geq 0$, the set of all functions, $f \in L^p(\Omega)$ with weak derivatives $D^\alpha f \in L^p(\Omega)$ up to the order $|\alpha| \leq m$. The sets $W^{m,p}(\Omega)$ are called Sobolev spaces.

**Lemma 1.6.** The Sobolev space $W^{m,p}(\Omega)$ equipped with the norm

$$\|f\|_{W^{m,p}(\Omega)} := \left(\int_{\Omega} \sum_{|\alpha| \leq m} |D^\alpha f(x)|^p \, dx \right)^{1/p} \quad \text{if } 1 \leq p < \infty$$

and

$$\|f\|_{W^{m,p}(\Omega)} := \max_{|\alpha| \leq m} \left(\text{ess sup}_{x \in \Omega} |D^\alpha f(x)|\right) \quad \text{if } p = \infty$$

is a Banach space. In particular, the Sobolev space $H^m(\Omega) := W^{m,2}(\Omega)$ is a Hilbert space with respect to the inner product

$$(u, v)_{H^m(\Omega)} := \sum_{|\alpha| \leq m} \langle D^\alpha u, D^\alpha v \rangle = \sum_{|\alpha| \leq m} \int_{\Omega} D^\alpha u \cdot D^\alpha v \, dx \quad \text{for all } u, v \in H^m(\Omega).$$
Proof. Here, we give the arguments for $1 \leq p < \infty$. For the case $p = \infty$, we refer to Adams (1975). The sets $W^{m,p}(\Omega)$ are real linear spaces (below we show that the sum $(f + g)$ of two elements $f, g \in W^{m,p}(\Omega)$ is an element of $W^{m,p}(\Omega)$). Next we have to show that the mapping $f \mapsto \|f\|_{W^{m,p}(\Omega)}$ is a norm, that is, the three conditions in Definition 1.1 hold. If $f = 0$, then $\|f\|_{W^{m,p}(\Omega)} = 0$ by definition of $\|\cdot\|_{W^{m,p}(\Omega)}$. Further, since $0 \leq \|f\|_{W^{m,p}(\Omega)} \leq \|f\|_{L^p(\Omega)}$ we conclude $f = 0$ in the sense of $L^p(\Omega)$, that is, functions which are equal up to a set of measure zero are identified. Thus, the first condition in Definition 1.1 holds. The second condition follows directly from the definition

$$
\|\lambda f\|_{W^{m,p}(\Omega)} = \left( \int_{\Omega} \left| \sum_{|\alpha| \leq m} D^\alpha \lambda f(x) \right|^p \, dx \right)^{1/p} = |\lambda| \left( \int_{\Omega} \left| \sum_{|\alpha| \leq m} D^\alpha f(x) \right|^p \, dx \right)^{1/p} = |\lambda| \|f\|_{W^{m,p}(\Omega)}.
$$

Let $f, g \in W^{m,p}(\Omega)$ and $|\alpha| \leq m$. The Minkowski’s inequality, Lemma 1.1, implies that $D^\alpha (f + g) = D^\alpha f + D^\alpha g \in L^p(\Omega)$ and

$$
\|D^\alpha (f + g)\|_{L^p(\Omega)} \leq \|D^\alpha f\|_{L^p(\Omega)} + \|D^\alpha g\|_{L^p(\Omega)}.
$$

Using the Minkowski’s inequality for sums, we get

$$
\|f + g\|^p_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} \|D^\alpha (f + g)\|^p_{L^p(\Omega)} \leq \left( \sum_{|\alpha| \leq m} \|D^\alpha f\|^p_{L^p(\Omega)} + \sum_{|\alpha| \leq m} \|D^\alpha g\|^p_{L^p(\Omega)} \right)^{1/p} \leq \left[ \left( \sum_{|\alpha| \leq m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{1/p} + \left( \sum_{|\alpha| \leq m} \|D^\alpha g\|^p_{L^p(\Omega)} \right)^{1/p} \right]^{1/p}
$$

from which the third condition follows. It remains to prove the completeness of the normed space $W^{m,p}(\Omega)$. Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $W^{m,p}(\Omega)$. Then, $(D^\alpha f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega)$ for any multi-index $\alpha$ with $|\alpha| \leq m$. Since $L^p(\Omega)$ is a Banach space, there is $f^\alpha \in L^p(\Omega)$ with $D^\alpha f_n \rightharpoonup f^\alpha$ in $L^p(\Omega)$. It remains to show that $f^\alpha = D^\alpha f$ where $f = f^0$. We show this for $\alpha = (1,0,\ldots,0)$. For any $\varphi \in C_0^\infty(\Omega)$ we have

$$
\left| \int_{\Omega} (f^\alpha \varphi + D^\alpha \varphi) \, dx \right| \leq \int_{\Omega} |f^\alpha - D^\alpha f_n| |\varphi| \, dx + \int_{\Omega} (D^\alpha f_n \varphi + f_n D^\alpha \varphi) \, dx 
\leq \int_{\Omega} |f - f_n| |\varphi| \, dx.
$$

The second term on the right hand side equals zero since $D^\alpha f_n$ is the weak derivative of $f_n$. The convergence of $f_n$ to $f$ and $D^\alpha f_n$ to $f^\alpha$ in $L^1(\Omega)$, respectively, imply the convergence in $L^1(\Omega)$, thus the first and third term on the right hand side tend to zero for $n \to \infty$. The left hand side
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is non-negative and independent of $n$, consequently we conclude that it is zero, which means $f^n$ is the weak derivative of $f$.

The properties for the inner product in $H^m(\Omega)$, see Definition 1.9, follow from its definition and the observation that $(\nu, \nu)_{H^m(\Omega)} = \|\nu\|^2_{H^m(\Omega)}$ for all $\nu \in W^{m,2}(\Omega)$.

**Definition 1.16.** We denote by $W^{m,p}_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in the norm $\| \cdot \|_{W^{m,p}(\Omega)}$. Analogously, $H^m_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $H^m(\Omega)$.

In the following, for $f \in W^{m,p}(\omega)$, $m \geq 0$, $1 \leq p \leq \infty$, the norm will be shortly denoted by $\|f\|_{m,p,\omega} := \|f\|_{W^{m,p}(\omega)}$ and a seminorm is introduced by

$$|f|_{m,p,\omega} := \left( \int_\omega \sum_{|\alpha|=m} |D^\alpha f(x)|^p \, dx \right)^{1/p} \quad \text{if } 1 \leq p < \infty$$

and

$$|f|_{m,p,\omega} := \max_{|\alpha|=m} \left( \text{ess sup}_{x \in \Omega} |D^\alpha f(x)| \right) \quad \text{if } p = \infty.$$ 

In case, $\omega = \Omega$ and/or $p = 2$ we omit $\Omega$ and $p$, respectively. Thus,

$$\|f\|_m = \|f\|_{2,m,\Omega}, \quad \|f\|_{m,p} = \|f\|_{2,m,p,\Omega}, \quad |f|_m = |f|_{2,m,\Omega}, \quad |f|_{m,p} = |f|_{2,m,p,\Omega}.$$

**Lemma 1.7 (Poincaré inequality).** There is a positive constant $c_p \leq 2a$ such that

$$\|\nu\|_{0,p} \leq c_p \|\nu\|_{1,p} \quad \text{for all } \nu \in W^{1,p}_0(\Omega).$$

Here, ‘$a$’ denotes the diameter of $\Omega$ in an arbitrary but fixed direction.

**Proof.** Suppose the Poincaré inequality is true for functions from $C^\infty_0(\Omega)$. Then, for any $\nu \in W^{1,p}_0(\Omega)$ there is a sequence $\nu_n \in C^\infty_0(\Omega)$ with $\nu_n \to \nu$ in $W^{1,p}_0(\Omega)$. We have

$$\|\nu\|_{0,p} \leq \|\nu - \nu_n\|_{0,p} + \|\nu_n\|_{0,p} \leq \|\nu - \nu_n\|_{0,p} + c_p \|\nu_n\|_{1,p} \leq (1 + c_p)\|\nu - \nu_n\|_{1,p} + c_p \|\nu\|_{1,p}.$$

Since $\|\nu - \nu_n\|_{1,p}$ tends to zero for $n \to \infty$, the Poincaré inequality is true for functions from $W^{1,p}_0(\Omega)$.

In order to prove the Poincaré inequality for functions $\nu \in C^\infty_0(\Omega)$ we extend $\nu : \Omega \to \mathbb{R}$ by setting $\nu(x) = 0$ for all $x \in \mathbb{R}^d \setminus \Omega$. Let $\Omega \subset [-a,+a] \times \mathbb{R}^{d-1}$, then

$$\nu(x) = \int_{-a}^{x_1} \frac{\partial \nu}{\partial x_1}(s,x_2,\ldots,x_d) \, ds$$
and Hölder’s inequality with $1/p + 1/q = 1$ implies
\[ |v(x)| \leq \left( \int_a^b \left| \frac{\partial v}{\partial x_1}(s,x_2,\ldots,x_d) \right|^{p} ds \right)^{1/p} \left( \int_a^b 1^{q} ds \right)^{1/q}, \]
\[ |v(x)|^p \leq (2a)^{p/q} \int_a^b \left| \frac{\partial v}{\partial x_1}(s,x_2,\ldots,x_d) \right|^p ds, \]
\[ \int_a^b |v(x)|^p dx_1 \leq (2a)^{1+p/q} \int_a^b \left| \frac{\partial v}{\partial x_1}(s,x_2,\ldots,x_d) \right|^p ds, \]
\[ \int_{\mathbb{R}^d} |v(x)|^p dx \leq (2a)^p \int_{\mathbb{R}^d} \left| \frac{\partial v}{\partial x_1}(x) \right|^p dx. \]

Using $v(x) = 0$ for $x \in \mathbb{R}^d \setminus \Omega$ the inequality follows.

The Poincaré inequality allows to show that the seminorm $| \cdot |_{m,p}$ is equivalent to the norm $\| \cdot \|_{m,p}$ on $W_{0}^{m,p}(\Omega)$. Let $v \in W_{0}^{m,p}(\Omega)$, then $D^\alpha v \in W_{1}^{0}(\Omega)$ for $|\alpha| \leq m-1$. We apply successively the inequality
\[ |D^\alpha v|_{0,p} \leq c_{\alpha} |D^\alpha v|_{1,p} \quad \text{for } |\alpha| = 0,1,\ldots,m-1 \]
to estimate the lower derivatives by the highest derivative and obtain
\[ |v|_{m,p} \leq \| v \|_{m,p} = \left( \sum_{|\alpha| \leq m} |D^\alpha v|_{0,p}^p \right)^{1/p} \leq C|v|_{m,p}, \]
that is, the equivalence of the seminorm to the norm.

An important tool will be the Sobolev embedding theorems which state that functions from $W^{m,p}(\Omega)$ belong to $W^{m-1,p'}$ for some $p' > p$. Moreover, it can be shown that for suitable numbers $m, p$ functions from $W^{m,p}(\Omega)$ belong to classical spaces of continuous and continuously differentiable functions, respectively.

**Definition 1.17.** We introduce the space of Hölder continuous functions with Hölder exponent $\beta \in (0,1)$
\[ C^{0,\beta}(\Omega) := \left\{ u \in C^0(\Omega) : \sup_{x,y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\beta} < \infty \right\} \]
equipped with the norm
\[ \| v \|_{C^{0,\beta}(\Omega)} = \sup_{x \in \Omega} |u(x)| + \sup_{x,y \in \Omega, x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\beta}. \]
Hölder continuous functions with a Hölder exponent $\beta = 1$ are also called Lipschitz continuous.

**Exercise 1.1.** Consider $f : [-1, +1] \to \mathbb{R}$ given by $f(x) = \sqrt{|x|}$. Prove that $f \in C^{0,1/2}([-1, +1])$ and that there is no $\beta \in (1/2, +1]$ with $f \in C^{0,\beta}([-1, +1])$. 
