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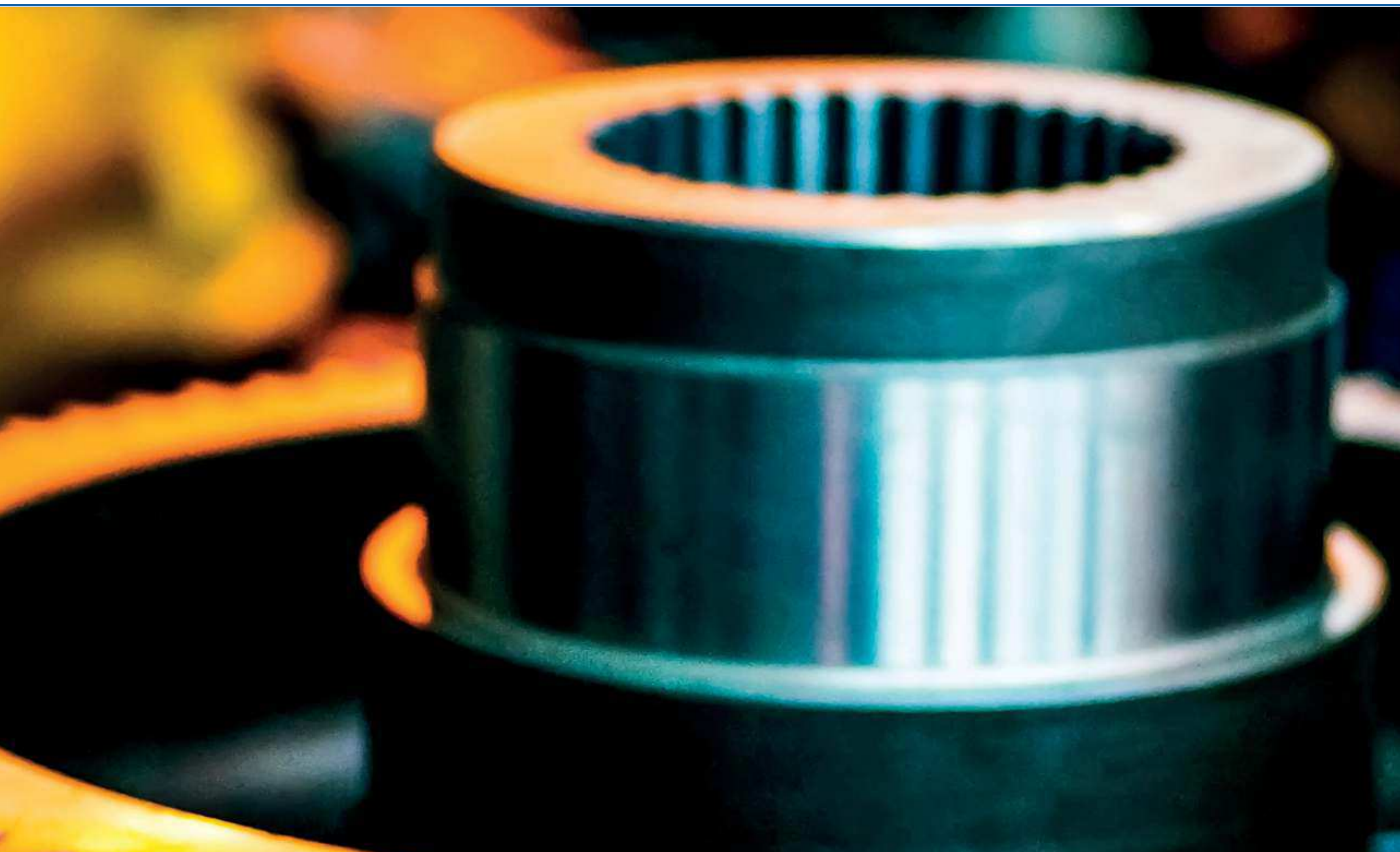

Chapter 1
Functions

This section will show you how to:

- understand and use the terms: function, domain, range (image set), one-one function, inverse function and composition of functions
- use the notation $f(x) = 2x^3 + 5$, $f : x \mapsto 5x - 3$, $f^{-1}(x)$ and $f^2(x)$
- understand the relationship between $y = f(x)$ and $y = |f(x)|$
- solve graphically or algebraically equations of the type $|ax + b| = c$ and $|ax + b| = cx + d$
- explain in words why a given function is a function or why it does not have an inverse
- find the inverse of a one-one function and form composite functions
- use sketch graphs to show the relationship between a function and its inverse.

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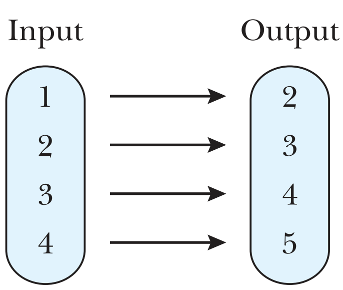
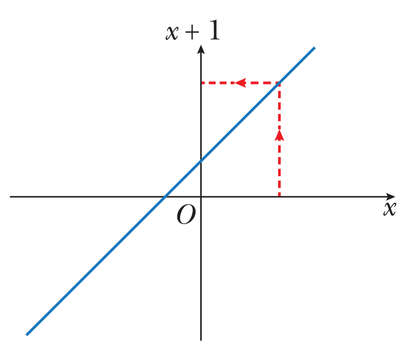
1.1 Mappings



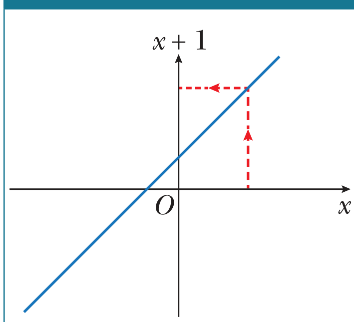
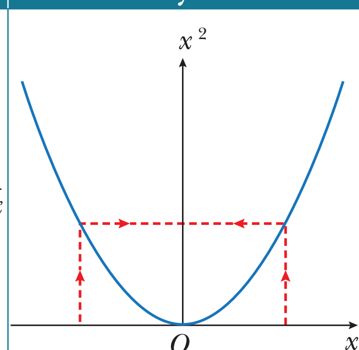
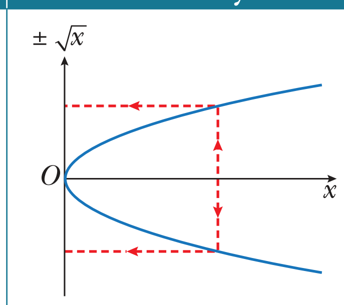
is called a **mapping diagram**.

The rule connecting the input and output values can be written algebraically as: $x \mapsto x + 1$.
This is read as 'x is mapped to $x + 1$'.



The mapping can be represented graphically by plotting values of $x + 1$ against values of x.

The diagram shows that for one input value there is just one output value.
It is called a **one-one** mapping.
The table below shows one-one, many-one and one-many mappings.

one-one	many-one	one-many
		
For one input value there is just one output value.	For two input values there is one output value.	For one input value there are two output values.

Exercise 1.1

Determine whether each of these mappings is one-one, many-one or one-many.

1 $x \mapsto x + 1 \quad x \in \mathbb{R}$

2 $x \mapsto x^2 + 5 \quad x \in \mathbb{R}$

3 $x \mapsto x^3 \quad x \in \mathbb{R}$

4 $x \mapsto 2^x \quad x \in \mathbb{R}$

5 $x \mapsto \dfrac{1}{x} \quad x \in \mathbb{R},\ x > 0$

6 $x \mapsto x^2 + 1 \quad x \in \mathbb{R},\ x \geqslant 0$

7 $x \mapsto \dfrac{12}{x} \quad x \in \mathbb{R},\ x > 0$

8 $x \mapsto \pm x \quad x \in \mathbb{R},\ x \geqslant 0$

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1.2 Definition of a function

A **function** is a rule that maps each x value to just one y value for a defined set of input values.

This means that mappings that are either $\begin{cases}\text{one-one} \\ \text{many-one}\end{cases}$ are called functions.

The mapping $x \mapsto x + 1$ where $x \in \mathbb{R}$, is a one-one function.

It can be written as $\begin{cases} \mathrm{f} : x \mapsto x + 1 & x \in \mathbb{R} \\ \mathrm{f}(x) = x + 1 & x \in \mathbb{R} \end{cases}$

($\mathrm{f} : x \mapsto x + 1$ is read as 'the function f is such that x is mapped to $x + 1$')

$\mathrm{f}(x)$ represents the output values for the function f.

So when $\mathrm{f}(x) = x + 1$, $\mathrm{f}(2) = 2 + 1 = 3$.

The set of input values for a function is called the **domain** of the function.

The set of output values for a function is called the **range** (or image set) of the function.

WORKED EXAMPLE 1

$\mathrm{f}(x) = 2x - 1 \quad x \in \mathbb{R}, -1 \leqslant x \leqslant 3$

a Write down the domain of the function f.

b Sketch the graph of the function f.

c Write down the range of the function f.

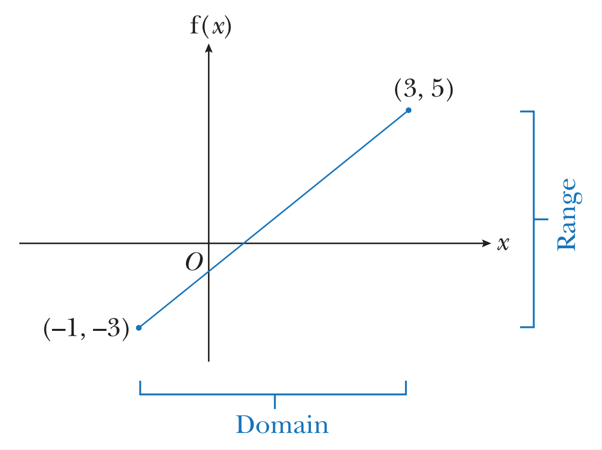
Answers

a The domain is $-1 \leqslant x \leqslant 3$.

b The graph of $y = 2x - 1$ has gradient 2 and a y-intercept of -1.

When $x = -1$, $y = 2(-1) - 1 = -3$

When $x = 3$, $y = 2(3) - 1 = 5$



c The range is $-3 \leqslant \mathrm{f}(x) \leqslant 5$.

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WORKED EXAMPLE 2

The function f is defined by $f(x) = (x - 2)^2 + 3$ for $0 \leqslant x \leqslant 6$.
Sketch the graph of the function.
Find the range of f.

Answers

$f(x) = (x - 2)^2 + 3$ is a positive quadratic function so the graph will be of the form $\cup$

$(x - 2)^2 + 3$

This part of the expression is a square so it will always be $\geqslant 0$.
The smallest value it can be is 0. This occurs when $x = 2$.

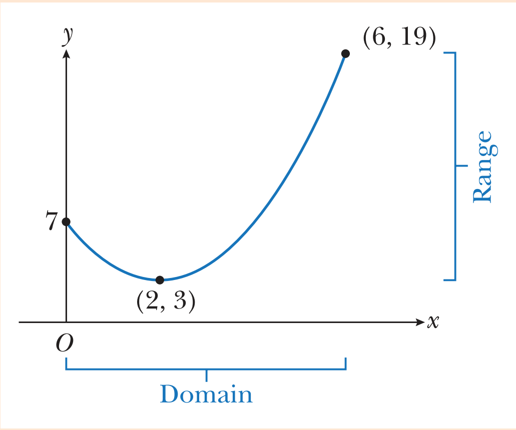
The minimum value of the expression is $0 + 3 = 3$ and this minimum occurs when $x = 2$.

So the function $f(x) = (x - 2)^2 + 3$ will have a minimum point at the point $(2, 3)$.

When $x = 0$, $y = (0 - 2)^2 + 3 = 7$.

When $x = 6$, $y = (6 - 2)^2 + 3 = 19$.

The range is $3 \leqslant f(x) \leqslant 19$.



Exercise 1.2

1 Which of the mappings in **Exercise 1.1** are functions?

2 Find the range for each of these functions.

a $f(x) = x - 5$, $\quad -2 \leqslant x \leqslant 7$

b $f(x) = 3x + 2$, $\quad 0 \leqslant x \leqslant 5$

c $f(x) = 7 - 2x$, $\quad -1 \leqslant x \leqslant 4$

d $f(x) = x^2$, $\quad -3 \leqslant x \leqslant 3$

e $f(x) = 2^x$, $\quad -3 \leqslant x \leqslant 3$

f $f(x) = \frac{1}{x}$, $\quad 1 \leqslant x \leqslant 5$

3 The function g is defined as $g(x) = x^2 + 2$ for $x \geqslant 0$.
Write down the range of g.

4 The function f is defined by $f(x) = x^2 - 4$ for $x \in \mathbb{R}$.
Find the range of f.

5 The function f is defined by $f(x) = (x - 1)^2 + 5$ for $x \geqslant 1$.
Find the range of f.

6 The function f is defined by $f(x) = (2x + 1)^2 - 5$ for $x \geqslant -\frac{1}{2}$.
Find the range of f.

7 The function f is defined by $f : x \mapsto 10 - (x - 3)^2$ for $2 \leqslant x \leqslant 7$.
Find the range of f.

8 The function f is defined by $f(x) = 3 + \sqrt{x - 2}$ for $x \geqslant 2$.
Find the range of f.

1.3 Composite functions

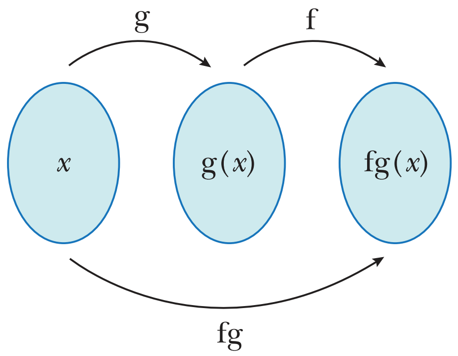
Most functions that you meet are combinations of two or more functions.

For example, the function $x \mapsto 2x + 5$ is the function 'multiply by 2 and then add 5'. It is a combination of the two functions g and f where:

$g : x \mapsto 2x$ (the function 'multiply by 2')

$f : x \mapsto x + 5$ (the function 'add 5')

So, $x \mapsto 2x + 5$ is the function 'first do g then do f'.



When one function is followed by another function, the resulting function is called a **composite function**.

fg(x) means the function g acts on x first, then f acts on the result.

Note:
$f^2(x)$ means ff(x), so you apply the function f twice.

WORKED EXAMPLE 3

The function f is defined by $f(x) = (x - 2)^2 - 3$ for $x > -2$.

The function g is defined by $g(x) = \dfrac{2x + 6}{x - 2}$ for $x > 2$.

Find fg(7).

Answers

fg(7)	g acts on 7 first and $g(7) = \dfrac{2(7) + 6}{7 - 2} = 4$
$= f(4)$	f is the function 'take 2, square and then take 3'
$= (4 - 2)^2 - 3$	
$= 1$	

WORKED EXAMPLE 4

$f(x) = 2x - 1$ for $x \in \mathbb{R}$ $\qquad$ $g(x) = x^2 + 5$ for $x \in \mathbb{R}$

Find **a** $fg(x)$ **b** $gf(x)$ **c** $f^2(x)$.

Answers

a $fg(x)$ — g acts on x first and $g(x) = x^2 + 5$

$= f(x^2 + 5)$ — f is the function 'double and take 1'

$= 2(x^2 + 5) - 1$

$= 2x^2 + 9$

b $gf(x)$ — f acts on x first and $f(x) = 2x - 1$

$= g(2x - 1)$ — g is the function 'square and add 5'

$= (2x - 1)^2 + 5$ — expand brackets

$= 4x^2 - 4x + 1 + 5$

$= 4x^2 - 4x + 6$

c $f^2(x)$ — $f^2(x)$ means $ff(x)$

$= ff(x)$ — f acts on x first and $f(x) = 2x - 1$

$= f(2x - 1)$ — f is the function 'double and take 1'

$= 2(2x - 1) - 1$

$= 4x - 3$

Exercise 1.3

1 $f : x \mapsto 2x + 3$ for $x \in \mathbb{R}$

$g : x \mapsto x^2 - 1$ for $x \in \mathbb{R}$

Find fg(2).

2 $f(x) = x^2 - 1$ for $x \in \mathbb{R}$

$g(x) = 2x + 3$ for $x \in \mathbb{R}$

Find the value of gf(5).

3 $f(x) = (x + 2)^2 - 1$ for $x \in \mathbb{R}$

Find $f^2(3)$.

4 The function f is defined by $f(x) = 1 + \sqrt{x - 2}$ for $x \geqslant 2$.

The function g is defined by $g(x) = \frac{10}{x} - 1$ for $x > 0$.

Find gf(18).

5 The function f is defined by $f(x) = (x - 1)^2 + 3$ for $x > -1$.

The function g is defined by $g(x) = \frac{2x + 4}{x - 5}$ for $x > 5$.

Find fg(7).

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6 $\text{h} : x \mapsto x + 2$ for $x > 0$

$\text{k} : x \mapsto \sqrt{x}$ for $x > 0$

Express each of the following in terms of h and k.

a $x \mapsto \sqrt{x} + 2$ **b** $x \mapsto \sqrt{x + 2}$

7 The function f is defined by $\text{f} : x \mapsto 3x + 1$ for $x \in \mathbb{R}$.

The function g is defined by $\text{g} : x \mapsto \dfrac{10}{2 - x}$ for $x \neq 2$.

Solve the equation $\text{gf}(x) = 5$.

8 $\text{g}(x) = x^2 + 2$ for $x \in \mathbb{R}$

$\text{h}(x) = 3x - 5$ for $x \in \mathbb{R}$

Solve the equation $\text{gh}(x) = 51$.

9 $\text{f}(x) = x^2 - 3$ for $x > 0$

$\text{g}(x) = \dfrac{3}{x}$ for $x > 0$

Solve the equation $\text{fg}(x) = 13$.

10 The function f is defined, for $x \in \mathbb{R}$, by $\text{f} : x \mapsto \dfrac{3x + 5}{x - 2}$, $x \neq 2$.

The function g is defined, for $x \in \mathbb{R}$, by $\text{g} : x \mapsto \dfrac{x - 1}{2}$.

Solve the equation $\text{gf}(x) = 12$.

11 $\text{f}(x) = (x + 4)^2 + 3$ for $x > 0$

$\text{g}(x) = \dfrac{10}{x}$ for $x > 0$

Solve the equation $\text{fg}(x) = 39$.

12 The function g is defined by $\text{g}(x) = x^2 - 1$ for $x \geqslant 0$.

The function h is defined by $\text{h}(x) = 2x - 7$ for $x \geqslant 0$.

Solve the equation $\text{gh}(x) = 0$.

13 The function f is defined by $\text{f} : x \mapsto x^3$ for $x \in \mathbb{R}$.

The function g is defined by $\text{g} : x \mapsto x - 1$ for $x \in \mathbb{R}$.

Express each of the following as a composite function, using only f and/or g:

a $x \mapsto (x - 1)^3$ **b** $x \mapsto x^3 - 1$ **c** $x \mapsto x - 2$ **d** $x \mapsto x^9$

1.4 Modulus functions

The **modulus** of a number is the magnitude of the number without a sign attached.

The modulus of 4 is written $|4|$.

$|4| = 4$ and $|-4| = 4$

It is important to note that the modulus of any number (positive or negative) is always a positive number.

The modulus of a number is also called the **absolute value**.

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The modulus of x, written as $|x|$, is defined as:

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

CLASS DISCUSSION

Ali says that these are all rules for absolute values:

$$|x+y| = |x| + |y| \qquad |x-y| = |x| - |y|$$

$$|xy| = |x| \times |y| \qquad \left|\frac{x}{y}\right| = \frac{|x|}{|y|} \qquad (|x|)^2 = x^2$$

Discuss each of these statements with your classmates and decide if they are:

Always true	Sometimes true	Never true

You must justify your decisions.

The statement $|x| = k$, where $k \geqslant 0$, means that $x = k$ or $x = -k$.

This property is used to solve equations that involve modulus functions.

So, if you are solving equations of the form $|ax + b| = k$, you solve the equations

$ax + b = k$ and $ax + b = -k$

If you are solving harder equations of the form $|ax + b| = cx + d$, you solve the equations

$ax + b = cx + d$ and $ax + b = -(cx + d)$.

When solving these more complicated equations you must always check your answers to make sure that they satisfy the original equation.

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WORKED EXAMPLE 5

Solve.

a $|2x+1| = 5$ **b** $|4x-3| = x$ **c** $|x^2 - 10| = 6$ **d** $|x-3| = 2x$

Answers

a $|2x+1| = 5$

$2x+1 = 5$ or $2x+1 = -5$

$2x = 4$ $\quad$ $2x = -6$

$x = 2$ $\quad$ $x = -3$

CHECK: $|2 \times 2 + 1| = 5$ ✓ and $|2 \times -3 + 1| = 5$ ✓

Solution is: $x = -3$ or 2.

b $|4x-3| = x$

$4x-3 = x$ or $4x-3 = -x$

$3x = 3$ $\quad$ $5x = 3$

$x = 1$ $\quad$ $x = 0.6$

CHECK: $|4 \times 0.6 - 3| = 0.6$ ✓ and $|4 \times 1 - 3| = 1$ ✓

Solution is: $x = 0.6$ or 1.

c $|x^2 - 10| = 6$

$x^2 - 10 = 6$ or $x^2 - 10 = -6$

$x^2 = 16$ $\quad$ $x^2 = 4$

$x = \pm 4$ $\quad$ $x = \pm 2$

CHECK: $|(-4)^2 - 10| = 6$ ✓, $|(-2)^2 - 10| = 6$ ✓, $|(2)^2 - 10| = 6$ ✓ and $|(4)^2 - 10| = 6$ ✓

Solution is: $x = -4, -2, 2$ or 4.

d $|x-3| = 2x$

$x-3 = 2x$ or $x-3 = -2x$

$x = -3$ $\quad$ $3x = 3$

$x = 1$

CHECK: $|-3-3| = 2 \times -3$ ✗ and $|1-3| = 2 \times 1$ ✓

Solution is: $x = 1$.

Exercise 1.4

1 Solve

a $|3x-2| = 10$ **b** $|2x+9| = 5$ **c** $|6-5x| = 2$

d $\left|\frac{x-1}{4}\right| = 6$ **e** $\left|\frac{2x+7}{3}\right| = 1$ **f** $\left|\frac{7-2x}{2}\right| = 4$

g $\left|\frac{x}{4} - 5\right| = 1$ **h** $\left|\frac{x+1}{2} + \frac{2x}{5}\right| = 4$ **i** $|2x-5| = x$

2 Solve

a $\left|\frac{2x-5}{x+3}\right| = 8$ **b** $\left|\frac{3x+2}{x+1}\right| = 2$ **c** $\left|1 + \frac{x+12}{x+4}\right| = 3$

d $|3x-5| = x+2$ **e** $x + |x-5| = 8$ **f** $9 - |1-x| = 2x$

3 Solve

a $|x^2 - 1| = 3$ **b** $|x^2 + 1| = 10$ **c** $|4 - x^2| = 2 - x$

d $|x^2 - 5x| = x$ **e** $|x^2 - 4| = x + 2$ **f** $|x^2 - 3| = x + 3$

g $|2x^2 + 1| = 3x$ **h** $|2x^2 - 3x| = 4 - x$ **i** $|x^2 - 7x + 6| = 6 - x$

4 Solve each of the following pairs of simultaneous equations

a $y = x + 4$, $y = |x^2 - 16|$ **b** $y = x$, $y = |3x - 2x^2|$ **c** $y = 3x$, $y = |2x^2 - 5|$

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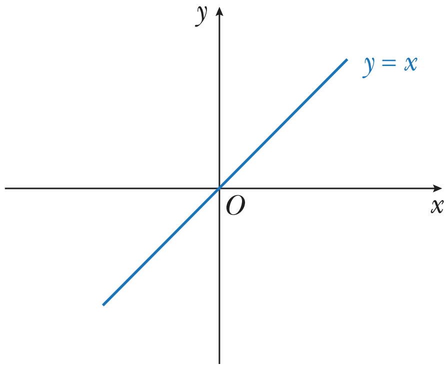
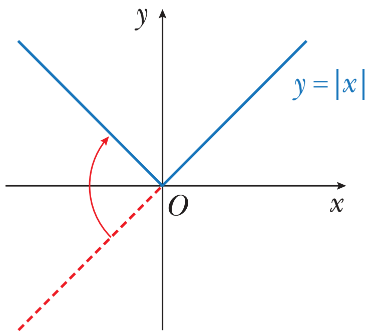
1.5 Graphs of $y = |f(x)|$ where f(x) is linear

Consider drawing the graph of $y = |x|$.

First draw the graph of $y = x$.



You then reflect in the x-axis the part of the line that is below the x-axis.



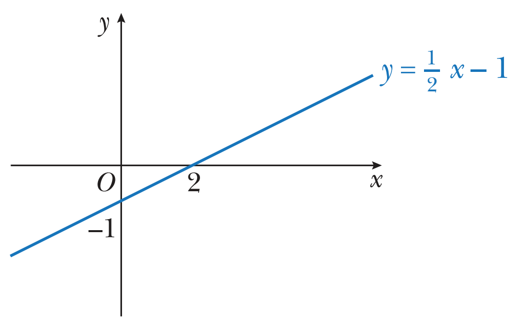
WORKED EXAMPLE 6

Sketch the graph of $y = \left|\frac{1}{2}x - 1\right|$, showing the coordinates of the points where the graph meets the axes.

Answers

First sketch the graph of $y = \frac{1}{2}x - 1$.

The line has gradient $\frac{1}{2}$ and a y-intercept of -1.



You then reflect in the x-axis the part of the line that is below the x-axis.