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Excerpt
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1 Classical Mechanics

Classical mechanics is the cornerstone of the GRE, making up 20% of the exam, and at the same time has the dubious distinction of being the subject that turns so many people *away* from physics. Your first physics class was undoubtedly a mechanics class, at which point you probably wondered what balls, springs, ramps, rods, and merry-go-rounds had to do in the slightest with the physics of the real world. So rather than (a) attempt the impossible task of covering your 1000-page freshman-year textbook in this much shorter reference, or (b) risk turning you away from physics before you've even taken the exam, we'll structure this chapter a little differently than the rest of the book. We're not going to review such things as Newton's laws, force balancing, or the definition of momentum; you should know these things in your sleep, or the rest of the exam will seem impossibly hard. Rather than review basic *topics*, we'll review standard *problem types* you're likely to encounter on the GRE. The more advanced topics will get their own brief treatment as well. After finishing this chapter, you will have reviewed nearly all the material you'll need for the classical mechanics section of the test, but in a format that is much more useful for the way the problems will likely be presented on the test. If you need a more detailed review of any of these topics, just open up any undergraduate physics text.

1.1 Blocks

One of the first things you learned in the first semester of freshman year physics was probably how to balance forces using free-body diagrams. Rather than rehash that discussion, which you can find in absolutely any textbook, we'll review it through a series of example problems that are GRE favorites. They involve objects, usually called "blocks," with certain masses, doing silly things like sitting on ramps, being pushed against springs, and traveling on carts. So here we go.

1.1.1 Blocks on Ramps

Here's a basic scenario (Fig. 1.1): a block of mass m is on a ramp inclined at an angle θ, and suppose we want to know the coefficient of static friction μ required to keep it in place. The usual solution method is to resolve any forces $\mathbf{F}$ into components along the ramp ($F_{\parallel}$) and perpendicular to the ramp ($F_{\perp}$). Rather than fuss with trigonometry or similar triangles, we can just do this by considering limiting cases, a theme that we'll return to throughout this book. In this case, we have to resolve the gravitational force $\mathbf{F}_g$. If the ramp is flat ($\theta = 0$), then there is no force in the direction of the ramp, so gravity acts entirely perpendicularly, and $F_{g,\parallel} = 0$. On the other hand, if the ramp is sheer vertical ($\theta = \pi/2$), then gravity acts entirely parallel to the ramp ($F_{g,\perp} = 0$), and the block falls straight down. Knowing that there must be sines and cosines involved, and the magnitude of $\mathbf{F}_g$ is mg, this uniquely fixes

$$F_{g,\parallel} = mg\sin\theta, \qquad F_{g,\perp} = mg\cos\theta.$$

For the block not to accelerate perpendicular to the ramp, we need the perpendicular forces to balance, which fixes the normal force to be $N = mg\cos\theta$. Then the frictional force is $F_f = \mu mg\cos\theta$, which must balance the component of gravity parallel to the ramp, $F_{g,\parallel} = mg\sin\theta$. Setting these equal gives

$$\mu mg\cos\theta = mg\sin\theta \implies \mu = \tan\theta.$$

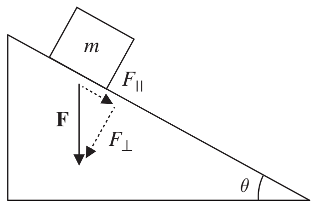


Figure 1.1 Free-body diagram of forces for a block on an inclined ramp.

Again, we can check this by limiting cases. If $\theta = 0$, then we don't need any friction to hold the block in place, and $\mu = 0$. If $\theta = \pi/2$, we need an infinite amount of friction to glue the block to the ramp and keep if from falling vertically, so $\mu = \infty$. Both of these check out.

Standard variants on this problem include applied forces and blocks attached to pulleys which hang over the side of the ramp, but surprisingly, neither the basic problem nor its variants have shown up on recent exams. Perhaps it is considered *too* standard by the GRE, such that most students will have memorized the problem and its variants so completely that it's not worth testing. In any case, consider it a simple review of how to resolve forces into components by using a limiting-cases argument, as this can potentially save you a lot of time on the exam.

1.1.2 Falling and Hanging Blocks

The next step up in complexity is to have two or more blocks interacting – for example, two blocks tied together with a rope, falling under the influence of gravity, or the same blocks hanging from a ceiling. These kinds of questions test your ability to identify precisely which forces are acting on which blocks. A foolproof, though time-consuming, method is to use *free-body diagrams*, where you draw each individual block and *only* the forces acting on it. This avoids the pitfall of double-counting, or applying the same force twice to two different objects, and ensures that you take into careful account the action/reaction balance of Newton's third law. See Example 1.1.

Sometimes, though, simple physical reasoning will suffice, especially in situations where the blocks aren't really distinct objects. For example, consider placing one block on top of another and letting them both fall under the influence of gravity. If we ignore air resistance, there is absolutely no physical distinction between the block–block system, and one larger block with the combined mass of both. In fact, a variant of precisely this argument was used in support of Galileo's discovery that the gravitational acceleration of objects was independent of their mass. We could even put a massless string between the two blocks, and the argument would *still* hold: since the whole system must fall with acceleration g, there can be no tension in the string. (Do the free-body analysis and check this yourself!) When interactions between the blocks become important, for example when they exert forces on one another through friction, then we must usually treat them as independent objects, though, as we'll see in Section 1.1.3, there are cases where the same kind of reasoning works.

EXAMPLE 1.1

A 5 kg block is tied to the bottom of a 20 kg block with a massless string. When an experimenter holds the 20 kg block stationary, the tension in the string is T_1. The experiment is repeated with the 20 kg block hanging under the 5 kg block, and the tension in the string is now T_2. What is T_2/T_1?

Our physical intuition tells us that $T_1/g = 5$ kg and $T_2/g = 20$ kg, since in both cases the function of the string is to support the weight of the lower block. So we expect $T_2/T_1 = 4$. This intuition is confirmed by a limiting-cases analysis: if the mass of the lower block is zero, then no matter the mass of the upper block, the string just dangles below the block with no tension, so the tension must be proportional to the mass of the lower block but independent of the mass of the upper one.

Let's check the intuition by doing a full free-body analysis. In order to treat both cases at once, call the mass of the top block m_1 and that of the bottom block m_2, as in Fig. 1.2. The forces on the two blocks are illustrated in Fig. 1.3. F is the force applied by the experimenter. Notice how the string tension acts up on the bottom block but down on the top block, and that the magnitude of T is the same for both blocks. For the purposes of the GRE, this is the *definition* of a massless string: it carries the same tension at every point. Setting the acceleration of m_2 equal to zero, since it is stationary, let's solve for T: $T - m_2g = 0$, so indeed, $T = m_2g$, the weight of the bottom block, and our intuition is correct. In this case it wasn't even necessary to consider the forces on the top block, a convenient time-saver!

EXAMPLE 1.1 (Cont.)

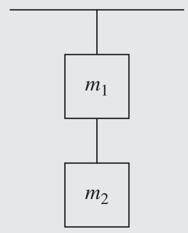


Figure 1.2 Two blocks suspended from one another by a massless string.

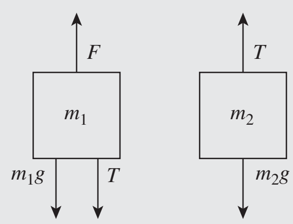


Figure 1.3 Free-body diagram for two blocks on a string.

1.1.3 Blocks in Contact

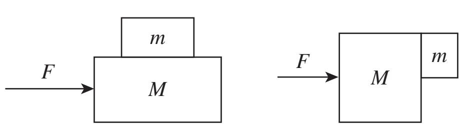
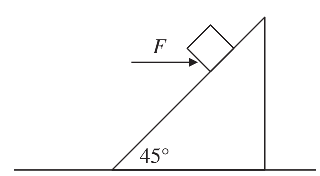


Figure 1.4 Typical setups for blocks moving together with friction.

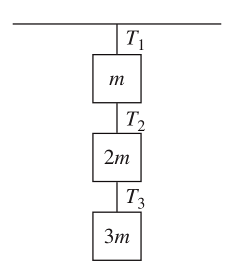
There are two standard setups for these kinds of problems, illustrated in Fig. 1.4. Both get at all the core concepts of force balancing, Newton's second and third laws, and friction. In the second setup, you might be asked, given friction between the two blocks, what the minimum force is such that the mass m does not fall down due to gravity, or, if m is placed on the surface as well, how the force of one block on another changes depending on whether F is applied to M or m. As with the falling and hanging blocks, the key is to remember that the blocks are *independent* objects, so we must consider the forces on each independently. See Example 1.2.

1.1.4 Problems: Blocks

1. A block of mass 5 kg is positioned on an inclined plane at angle 45°. A force of 10 N is applied to the block, parallel to the ground. If the coefficient of kinetic friction is 0.5, which of the following is closest to the acceleration of the block? Assume there is no static friction.



(A) $\sqrt{2}$ m/s^2 up the ramp
(B) $\sqrt{2}$ m/s^2 down the ramp
(C) $5\sqrt{2}$ m/s^2 up the ramp
(D) $5\sqrt{2}$ m/s^2 down the ramp
(E) $25\sqrt{2}$ m/s^2 down the ramp



2. Three blocks of masses m, $2m$, and $3m$ are suspended from the ceiling using ropes, as shown in the diagram. Which of the following correctly describes the tension in the three rope segments, labeled T_1, T_2, and T_3?

(A) $T_1 < T_2 < T_3$
(B) $T_1 < T_2 = T_3$

EXAMPLE 1.2

Here's an example using the setup shown in Fig. 1.4 (left): A block of mass 2 kg sits on top a block of mass 5 kg, which is placed on a frictionless surface. A force of 10 N is applied horizontally to the 5 kg block. What is the minimum coefficient of static friction between the two blocks such that they move together without slipping?

We could do a full free-body diagram of all the forces in the problem, but simple physical reasoning provides a useful shortcut. Note that, as long as the blocks don't slip, the two blocks are really behaving as one object of mass $M + m$, just like the falling blocks attached by a massless string in Section 1.1.2 above. Thus we expect the final expression for μ to depend on the combination $M + m$, rather than M or m individually, since μ determines whether the two blocks stick together and act as a composite system.

To see this explicitly, let's analyze the motion of the top block first. The forces on the top block are its weight $-mg$, the normal force N_1 provided by the bottom block, and the frictional force $F_f = \mu N_1$. Since the top block is not accelerating vertically, we must have $N_1 = mg$ and the net force forward is $F_f = \mu mg$. Now the top block will begin to slip just as the force F_1 on it is equal to the maximum force that friction can supply; in other words, the slipping condition is $F_1 = F_f = \mu mg$. But by definition we also know that $F_1 = ma$, where a here is the acceleration of the two-block system – since both blocks are stuck together, they experience the same acceleration. The mass of the total system is $M + m$ and the applied force is F, so $F = (M + m)a$. Substituting the values for a and F_1 into $F_1 = ma$, we find

$$\mu mg = m\frac{F}{M+m} \implies \mu = \frac{F}{(M+m)g},$$

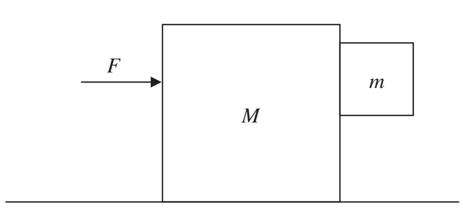
which as expected depends on $M + m$. Notice that we didn't ever have to do a free-body analysis of the second block alone: instead, we applied Newton's second law to the two-block system in the second step.

Of course, we can also do a free-body analysis for the block of mass M. We have the applied force F acting forwards, but there is also a force acting *backwards*, from Newton's third law: the bottom block is providing a frictional force which pushes the top block forwards, so the bottom block feels an equal force backwards. The net horizontal force is then $F - \mu mg$, where the second term is the magnitude of the friction force derived above. The acceleration of the bottom block is $a = \frac{1}{M}(F - \mu mg)$, and we want the frictional force on the top block to provide at least this acceleration, $a = F_f/m$, or the blocks will slip. Thus

$$\frac{1}{M}(F - \mu mg) = \frac{\mu mg}{m} \implies \mu = \frac{F}{(M+m)g},$$

the same answer as before. Plugging in the numbers, we find $\mu \approx 0.14$.

(C) $T_1 = T_2 = T_3$
(D) $T_1 = T_2 > T_3$
(E) $T_1 > T_2 > T_3$



3. Two blocks of masses M and m are oriented as shown in the diagram. The block M moves on a surface with coefficient of kinetic friction μ_1, and the coefficient of static friction between the two blocks is μ_2. What is the minimum force F which must be applied to M such that m remains stationary relative to M?

(A) $\frac{\mu_1}{\mu_2} mg$
(B) $\frac{\mu_1}{\mu_2}\frac{Mm}{M+m} g$
(C) $(\mu_1 M + \mu_2 m)\, g$
(D) $\left(\mu_1 + \frac{1}{\mu_2}\right)(m+M)g$
(E) $\left(\mu_1 M + \frac{m}{\mu_2}\right) g$

1.2 Kinematics

Kinematics is the first physics that almost everyone learns, so it should be burned into the reader's mind already. For almost all problems it is sufficient to know the equations of motion for a particle undergoing constant acceleration. The primary types of problem worth reviewing are projectile motion problems and problems involving reference frames. To solve projectile motion in two dimensions, you only need the equations of motion for the x- and y-coordinates of the particle,[1]

$$x(t) = v_{0x}t + x_0, \qquad y(t) = -\frac{1}{2}gt^2 + v_{0y}t + y_0, \tag{1.1}$$

where we define coordinates such that gravity acts in the negative y-direction and $g = 10$ m/s^2. Restricting to one dimension, there is another useful formula relating the initial and final velocities of an object, v_i and v_f, its acceleration a, and the change in position between the initial and final states Δx, if the acceleration is constant:

$$v_f^2 - v_i^2 = 2a\Delta x. \tag{1.2}$$

A two-line derivation of this formula uses the work–energy theorem, reviewed in Section 1.3.4.

For problems involving reference frames, just solve the problem in one frame, and then transform to the frame that the problem is asking about. For example, consider the situation in Fig. 1.5: a ball is thrown out of a car moving at constant velocity. Ignoring air resistance, in the frame of the car, the ball moves directly perpendicular to the road. In the frame of an observer at rest, the car is moving forwards, so the motion of the ball is the sum of the two velocities. In other words, the ball moves diagonally, both forward and away from the road. See Example 1.3.

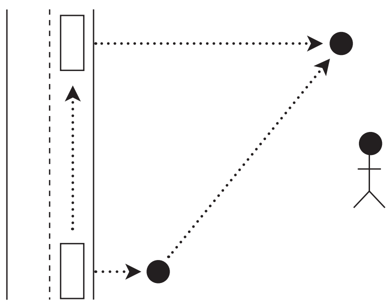


Figure 1.5 A ball thrown out of a moving car, in the frame of a stationary bystander.

From the point of view of solving problems, however, one should avoid kinematics like the plague. It often results in having to solve quadratic equations, and although this is simple in principle, it is usually a huge waste of time. As a rule of thumb, only resort to kinematics if you need to know the explicit time dependence of a system. In nearly all other cases, the basic energy considerations discussed in Section 1.3 will be faster and computationally simpler.

1.2.1 Circular Motion

One kinematic situation that arises often on GRE questions is circular motion. We will consider this in slightly more detail in Section 1.6 when we discuss orbits. For now, consider a particle moving on a circular path. Its acceleration vector can always be decomposed into radial and tangential components. If its tangential acceleration is zero, then its tangential velocity is constant; it is moving in *uniform circular motion* about the center of the circle. But its radial acceleration is nonzero, and has value

$$a = \frac{v^2}{r}, \tag{1.3}$$

where v is the speed of the particle and r is the radius of its orbit. From this, we can immediately also infer that the force needed to keep the particle in its orbit, the *centripetal force*, is

$$F = \frac{mv^2}{r}. \tag{1.4}$$

Indeed, since the tangential acceleration is zero, it must experience some force, directed radially inwards, that keeps it moving in a circular path at a constant speed. Remember that this does not tell you what *kind* of force is acting on the body. It just tells you that if you see a body moving uniformly in a circle of radius r with *constant* speed v, then you can determine what centripetal force must be acting on it.

While uniform circular motion is perhaps the most common example, it is certainly not the most general. There are many cases of nonuniform circular motion: for example, a roller-coaster going around a circular loop-the-loop, or a vertical pendulum attached to a rigid rod with sufficient initial speed to complete a full revolution. In these cases the angle between the gravitational force vector and the velocity vector varies as the object goes around the circle, giving a varying tangential acceleration in addition to the centripetal force, and the above formulas do not apply throughout the whole orbit. However, the uniform circular motion equations *do* apply

[1] In this book, we use the convention of numbering *only equations describing general results worth memorizing for the exam*. We therefore numbered the kinematics formulas here, while we didn't number the equations in the previous section that applied to a specific problem involving blocks. This should help you focus on remembering the equations that actually matter for the exam. We have listed all numbered equations in the equation index at the back of the book, along with page numbers, for your convenience.

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EXAMPLE 1.3

Suppose an astronaut is on a rocket that is moving vertically at constant speed u. When the rocket is at a height h, the astronaut throws a ball horizontally out of the rocket with velocity w, as shown in Fig. 1.6. What is the speed of the ball when it hits the ground?

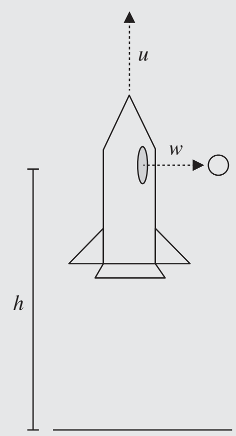

Figure 1.6 A ball is thrown horizontally at velocity w out of a rocket moving vertically upwards at constant velocity u.

In the frame of the rocket, the ball's initial y-velocity is zero, but in the ground frame its initial velocity is u, the relative velocity between the two reference frames. From our kinematic formula (1.2) above, we have for the y-component of the velocity

$$v_y = \sqrt{u^2 + 2gh}.$$

The x-component of the velocity is always the same, $v_x = w$, since no forces act in the x-direction, so we have a total speed

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{u^2 + w^2 + 2gh}.$$

at two very special places: the top and bottom of the circle, where gravity acts purely vertically, and thus radially, such that the object is instantaneously in uniform circular motion. At all other points in the orbit, other methods (such as energy conservation) must be used to find the velocity.

The centripetal force equation is not so interesting on its own, so a very common class of problems involves combining it with some other type of physics. A typical template might look roughly like this: A particle is moving in a circle. Identify the physics that is causing the centripetal force. Set the expression for this force equal to the centripetal force. Then solve for whatever quantity is requested. See Example 1.4.

1.2.2 Problems: Kinematics

1. A cannonball is fired with a velocity v. At what angle from the ground must the cannonball be fired in order for it to hit an enemy that is at the same elevation, but a distance d away?

   (A) $\arcsin(v/gd)$
   (B) $\arcsin(gd/(2v))$
   (C) $\arcsin(2gd/v)$
   (D) $(1/2)\arcsin(gd/v^2)$
   (E) $\arcsin(gd/v^2)$

2. A satellite (mass m) is in geosynchronous orbit around the Earth (mass M_E), such that its orbit has the same period as the Earth's rotation. If the Earth has angular rotational velocity ω, what is the radius of a geosynchronous orbit?

   (A) $\dfrac{GM_E}{\omega^2}$
   (B) $\dfrac{Gm}{\omega^2}$
   (C) $\left(\dfrac{GM_E}{\omega^2}\right)^{1/3}$
   (D) $\sqrt{\dfrac{GM_E}{\omega^2}}$
   (E) There is no possible geosynchronous orbit.

EXAMPLE 1.4

An electron (charge e) moves perpendicularly to a uniform magnetic field of magnitude B. If the kinetic energy of the particle doubles, by how much must the magnetic field change for the particle's trajectory to remain unchanged?

We know that the magnetic force on the electron is perpendicular to its motion (see Section 2.2.6 for a review), so it is a *centripetal* force, and the electron moves in a circle. More specifically, the forces are constant, so the electron executes uniform circular motion. Setting the magnetic and centripetal forces equal gives us

$$evB = \frac{mv^2}{r}.$$

Rearranging just a little, we can find the magnetic field

$$B = \frac{mv}{er}.$$

If the kinetic energy of the particle doubles, its velocity increases by $\sqrt{2}$, so B must increase to $\sqrt{2}B$ in order to maintain the same radius. This template occurs very frequently. Though circular motion can involve many different types of physics, identifying the centripetal force and setting it equal to mv^2/r will give you an additional equation to help solve the problem at hand.

1.3 Energy

Conservation of energy can be stated as follows:

> *If an object is acted on only by conservative forces, the sum of its kinetic and potential energies is constant along the object's path.*

Conservative forces are those for which the work done by the force is independent of the path taken between the starting and ending points, but the most useful definition (although it seems tautological) is *a force to which you can associate a (time-independent) potential energy*. The most common examples are gravity, spring forces, and electric forces. The most common example of a force that is *not* conservative, and probably the only such example you'll see on the GRE, is friction: an object traveling from point A to B and back to A will slow down due to friction the whole way through, even though the starting point is the same as the ending point.

A standard subset of GRE classical mechanics problems are most easily solved by straightforward application of conservation of energy. It's important to recognize these problems so you immediately jump to the fastest solution method, rather than fish around for the right kinematics formulas, so we'll state a general principle:

> *If you want to know how fast or how far something goes, use conservation of energy.*
>
> *If you want to know how much time something takes, use kinematics.*

It's baffling that this simple dichotomy isn't introduced in first-year physics courses. It's based on the idea that total energy is a combination of kinetic energies, which depend on velocities, and potential energies, which depend on positions. Setting $E_{\text{initial}} = E_{\text{final}}$ lets you solve for one in terms of the other, but nowhere in the equation does time appear explicitly. On the other hand, kinematics gives you explicit formulas for position and velocity as a function of time t (see equation (1.1)). Of course, some problems will require a combination of both methods, for example using conservation of energy to solve for a velocity which you then plug into a kinematics formula, but, as a very general rule, if time doesn't appear in the problem then you can leave kinematics out of the picture. However, we'll address a common exception to this rule at the end of Section 1.3.2.

1.3.1 Types of Energy

To begin with, you should know the following formulas *cold*:

Translational kinetic energy:	$\frac{1}{2}mv^2$	(1.5)
Rotational kinetic energy:	$\frac{1}{2}I\omega^2$	(1.6)
Gravitational potential energy on Earth:	mgh	(1.7)
Spring potential energy:	$\frac{1}{2}kx^2$	(1.8)

Hopefully the standard notation is familiar to you: v is linear velocity, ω is angular velocity, m is mass, I is the

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moment of inertia, h and x are displacements, g is gravitational acceleration at Earth's surface (which should *always* be approximated to 10 m/s^2 on the GRE when numerical computations are required), and k is the spring constant. There are two important points to remember about potential energy:

- It is only defined up to an additive constant: we are free to choose the zero of potential energy wherever is most convenient, which is usually some physically relevant position such as the bottom of a ramp or the uncompressed length of a spring.
- It is measured from the *center of mass* of an extended object. The usefulness of the center of mass concept (see Section 1.4.4) is that it allows us to treat extended objects like point masses, with all their mass concentrated at the location of the center of mass.

There are other types of potential energy, but all can be summarized by a definition. For any conservative force **F**, the change in potential energy ΔU between points a and b is

$$\Delta U = -\int_a^b \mathbf{F} \cdot d\mathbf{l}. \tag{1.9}$$

The line integral looks scary but it really isn't, since in all cases of interest the integral will be along the direction of the force vector. Probably the only time you might have to use this formula is if you can't remember the electrostatic or gravitational potential right away, so we'll do that example here. The gravitational force between two masses m_1 and m_2 is

$$\mathbf{F}_{\text{grav}} = \frac{Gm_1 m_2}{r^2}\hat{\mathbf{r}}. \tag{1.10}$$

You may have seen this equation in the form

$$\mathbf{F}_{\text{grav, 1 on 2}} = -\frac{Gm_1 m_2}{r^2}\hat{\mathbf{r}},$$

stating that the force on mass m_2 from m_1 points along the vector $\hat{\mathbf{r}}$ *from* m_1 *to* m_2, with the minus sign to indicate that the force is attractive. As we'll see, there are minus signs everywhere, so even though it's (deliberately) a bit ambiguous, we find (1.10) a more useful mnemonic for the GRE – just remember that gravity is attractive, and fill in the signs depending on which force (1 on 2 or 2 on 1) you're computing. See Example 1.5.

Alternatively, if you're given the potential, you can compute the force by inverting equation (1.9):

$$\mathbf{F} = -\nabla U. \tag{1.11}$$

Again, watch the minus sign!

1.3.2 Kinetic/Potential Problems

The simplest energy problem involves a mass on a ramp of some complicated shape, asking about its final velocity given that it starts at a certain height, or what initial height it will need to get over a loop-the-loop, or something like that. Because gravity is a conservative force, the shape of the ramp is irrelevant, as long as it's frictionless. If there's friction, then the shape of the ramp *does* matter because the work done by friction depends on the distance traveled – we'll get to that in a bit. First we'll look at a standard example.

EXAMPLE 1.5

Let's find the gravitational potential of a satellite of mass m in the gravitational field of the Earth, of mass M. The most common choice is to set the zero of potential energy at $r = \infty$, so the potential of the satellite at a finite distance r from the center of the Earth is

$$U(r) = -\int_\infty^r -\frac{GmM}{r'^2}\, dr' = -\left.\frac{GmM}{r'}\right|_\infty^r = -\frac{GmM}{r}.$$

Note the signs: the force on the satellite is directed *towards* the Earth, or in the $-\hat{\mathbf{r}}$ direction, but $d\mathbf{l} = +\hat{\mathbf{r}}\, dr'$, so the dot product is negative. The final sign makes sense because gravitational potential decreases (that is, becomes more negative) as the satellite gets closer to the Earth; in other words, it is attracted towards the Earth. Probably the most confusing part of this whole business is the signs, which the GRE *loves* to exploit. Rather than worrying about putting the signs in the right place throughout the whole problem, it may be best to just compute the unsigned quantity, then fill in the sign at the end with physical reasoning.

EXAMPLE 1.6

A block slides down a frictionless quarter-circle ramp of radius R, as shown in Fig. 1.7. How fast is it traveling when it reaches the bottom?

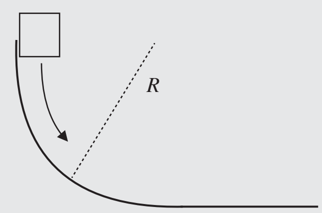

Figure 1.7 Block sliding down a quarter-circle ramp.

The quarter-circle shape is irrelevant except for the fact that it gives us the initial height: the block starts at height R above the bottom. At the top, the block is stationary, so its velocity is zero and there is no kinetic energy; all the energy is potential. Here the obvious choice is to set the zero of gravitational potential energy at the bottom of the ramp, so that the potential at the top is mgR. Wait a minute – the problem didn't tell us the mass of the block! Let's call it m, and see if we can resolve the situation as we finish the problem. At the bottom of the ramp, all the energy is kinetic, because we've defined the potential energy to be zero there. If the block's speed at the bottom is v, then its kinetic energy is $\frac{1}{2}mv^2$. We now apply conservation of energy:

$$0 + mgR = \frac{1}{2}mv^2 + 0$$
$$\implies v = \sqrt{2gR}.$$

Conveniently enough, the mass cancels out since both the kinetic and potential energies are directly proportional to m.

There are a couple things to note about Example 1.6:

- This was the very simplest version of the problem. The block could have had a nonzero speed at the top, in which case it would have had nonzero kinetic energy there. So don't automatically assume that conservation of energy is equivalent to "potential at top equals kinetic at bottom," which is *not* true in general!
- This problem can easily be extended to a kinematics problem by asking how far the block travels after it is launched off the bottom of the ramp, assuming the ramp is some height above the ground.[2] The first step of this problem would still be finding the initial velocity when it leaves the ramp, exactly as we found above.
- The fact that the mass cancels out is actually quite common in problems involving *only* a gravitational potential, since both kinetic and potential energies are proportional to m. So if the problem doesn't give you a mass, don't panic! That's actually a strong clue that the right approach is conservation of energy.

1.3.3 Rolling Without Slipping

A common variant of the above problem is a round object (sphere, cylinder, and so forth) rolling down a ramp. If the object *rolls without slipping*, then its linear velocity v and angular velocity ω are related by

$$v = R\omega, \tag{1.12}$$

where R is the radius. (Dimensional analysis dictates where to put the R so that v comes out with the correct units.) Then in addition to its kinetic energy, $\frac{1}{2}mv^2$, the object also has rotational kinetic energy $\frac{1}{2}I\omega^2$, where I is its moment of inertia. The rolling-without-slipping condition (1.12) lets

[2] Note that this is an exception to our rule about distances being associated with energy rather than kinematics: the block travels with constant horizontal speed once it leaves the ramp, so the only thing dictating how far it goes is the *time* it takes to fall vertically to the ground, which we must get from kinematics. So this is an exception only because it's actually a two-dimensional problem.

you substitute v for ω and express everything in terms of v, after which you can solve for v exactly as above. Incidentally, it's *friction* that causes rolling without slipping, as friction is responsible for resisting the motion of the point of contact with the object so that it can instantaneously rotate around this pivot. In this situation friction does no work, but instead is responsible for diverting translational energy into rotational energy. Without friction, all objects would simply slide, rather than roll.

Rolling-without-slipping problems almost always boil down to the kinds of cancellations shown in Example 1.7: the kinetic energy is of the form $\alpha m v^2$, with α some number that accounts for the moment of inertia. Here, α was 3/4 for the cylinder and 7/10 for the sphere. Notice that the problem didn't ask which object arrives *first*, only which object had the greater velocity at the bottom: the former is a kinematics question, which by our general principle can't be answered by conservation of energy alone.

EXAMPLE 1.7

A cylinder of mass m and radius r, and a sphere of mass M and radius R, both roll without slipping down an inclined plane from the same initial height h, as shown in Fig. 1.8. The cylinder arrives at the bottom with greater linear velocity than the sphere

(A) if $m > M$
(B) if $r > R$
(C) if $r > \frac{4}{5}R$
(D) never
(E) always

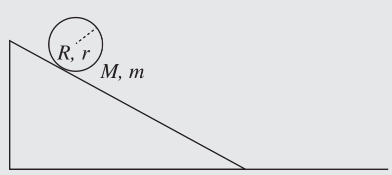

Figure 1.8 Ball or cylinder rolling down an inclined ramp.

You should immediately recognize that the mass is a red herring: since the moment of inertia is proportional to the mass, the same arguments as in Section 1.3.2 go through, and the mass cancels out of the conservation of energy equation for both objects. But let's see how this works explicitly. The moments of inertia are $\frac{1}{2}mr^2$ for the cylinder and $\frac{2}{5}MR^2$ for the sphere (neither of which you should memorize, since they're among the few useful quantities given in the table of information at the start of the test). The energy conservation equations read

$$mgh = \frac{1}{2}mv_{\text{cyl}}^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega_{\text{cyl}}^2 \qquad \text{(cylinder)},$$

$$Mgh = \frac{1}{2}Mv_{\text{sph}}^2 + \frac{1}{2}\left(\frac{2}{5}MR^2\right)\omega_{\text{sph}}^2 \qquad \text{(sphere)}.$$

As promised, we can cancel m from both sides of the first equation, and M from both sides of the second, which lets us equate the two right-hand sides. Now, substituting $\omega_{\text{cyl}} = v_{\text{cyl}}/r$ and $\omega_{\text{sph}} = v_{\text{sph}}/R$, we have

$$\left(\frac{1}{2}+\frac{1}{4}\right)v_{\text{cyl}}^2 = \left(\frac{1}{2}+\frac{1}{5}\right)v_{\text{sph}}^2.$$

The radii also cancel! So we can read off immediately that $v_{\text{cyl}} < v_{\text{sph}}$, and the cylinder *always* arrives slower, choice D.