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SCIENTIFIC PAPERS

BY

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VOLUME II
TIDAL FRICTION
AND
COSMOGONY

CAMBRIDGE:
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PREFACE

THE papers contained in the present volume form in effect a single investigation in speculative astronomy. The tidal oscillations of the mobile parts of a planet must be subject to frictional resistance, and this simple cause gives rise to a diversity of astronomical effects worthy of examination.

The earlier portion of the investigation was undertaken with the object of explaining, if possible, the obliquity of the earth's equator to the ecliptic, and the results attained were so fruitful and promising that it seemed well to examine the whole subject with the closest attention, and to discuss the various collateral points which arose in the course of the work.

It is the experience of every investigator that he reaches his goal by a devious path, and this, at least, has been the case in the present group of papers. If then the whole field were now to be retraversed, it is almost certain that the results might be obtained more shortly. Then, again, there is another cause which precludes brevity, for when an entirely new subject is under consideration every branch road must be examined with care. By far the greater number of the forks in the road lead only to blind alleys; and it is often impossible to foresee, at the cross roads, which will be the main highway, and which a blind alley. Clearness of view is only reached by the pioneer after much labour, and as he first passes along his path he has to grope his way in the dark without the help of any signpost.

This may be illustrated by what actually occurred to me, for when I first found the quartic equation (p. 102) which expresses the identity between the lengths of the day and of the month, I only regarded it as giving the configuration towards which the retrospective integration was leading back. I well remember thinking that it was just as well to find the other roots of the equation, although I had no suspicion that anything of interest would be discovered thereby. As of course I ought to have foreseen, the result threw a flood of light on the whole subject, for it showed how the system must have degraded, through loss of energy, from a configuration represented by the first real root to another represented by the second. Moreover the motion in the first configuration was found to be unstable whilst that in
the second was stable. Thus this quartic equation led to the remarkably simple and illuminating view of the theory of tidal friction contained in the fifth paper (p. 195); and yet all this arose from a point which appeared at first sight barely worth examining.

I wish now, after the lapse of more than twenty years, to avail myself of this opportunity of commenting on some portions of the work and of reviewing the theory as a whole.

The observations of Dr Hecker* and of others do not afford evidence of any considerable amount of retardation in the tidal oscillations of the solid earth, for, within the limits of error of observation, the phase of the oscillation appears to be the same as if the earth were purely elastic. Then again modern researches in the lunar theory show that the secular acceleration of the moon’s mean motion is so nearly explained by means of pure gravitation as to leave but a small residue to be referred to the effects of tidal friction. We are thus driven to believe that at present tidal friction is producing its inevitable effects with extreme slowness. But we need not therefore hold that the march of events was always so leisurely, and if the earth was ever wholly or in large part molten, it cannot have been the case.

In any case frictional resistance, whether it be much or little and whether applicable to the solid planet or to the superincumbent ocean, is a true cause of change, and it remains desirable that its effects should be investigated. Now for this end it was necessary to adopt some consistent theory of frictionally resisted tides, and the hypothesis of the earth’s viscosity afforded the only available theory of the kind. Thus the first paper in the present volume is devoted to the theory of the tides of a viscous spheroid. It may be that nothing material is added by solving the problem also for the case of elasto-viscosity, but it was well that that hypothesis should also be examined.

I had at a previous date endeavoured to determine the amount of modification to which Lord Kelvin’s theory of the tides of an elastic globe must be subject in consequence of the heterogeneity of the earth’s density, and this investigation is reproduced in the second paper. Dr Herglotz has also treated the problem by means of some laborious analysis, and finds the change due to heterogeneity somewhat greater than I had done. But we both base our conclusions on assumptions which seem to be beyond the reach of verification, and the probability of correctness in the results can only be estimated by means of the plausibility of the assumptions.

The differential equations which specify the rates of change in the various elements of the motions of the moon and the earth were found to be too

complex to admit of analytical integration, and it therefore became necessary to solve the problem numerically. It was intended to draw conclusions as to the history of the earth and moon, and accordingly the true values of the mass, size and speed of rotation of the earth were taken as the basis of computation. But the earth was necessarily treated as being homogeneous, and thus erroneous values were involved for the ellipticity, for the precessional constant and for the inequalities in the moon's motion due to the oblateness of the earth. It was not until the whole of the laborious integrations had been completed that it occurred to me that an appropriate change in the linear dimensions of the homogeneous earth might afford approximately correct values for every other element. Such a mechanically equivalent substitute for the earth is determined on p. 439, and if my integrations should ever be repeated I suggest that it would be advantageous to adopt the numerical values there specified as the foundation for the computations.

The third paper contains the investigation of the secular changes in the motions of the earth and moon, due to tidal friction, when the lunar orbit is treated as circular and coincident with the ecliptic. The differential equations are obtained by means of the disturbing forces, but the method of the disturbing function is much more elegant. The latter method is used in the sixth paper (p. 208), which is devoted especially to finding the changes in the eccentricity and the inclination of the orbit. However the analysis is so complicated that I do not regret having obtained the equations in two independent ways. As the sixth paper was intended to be supplementary to the third, the disturbing function is developed with the special object of finding the equations for the eccentricity and the inclination, but an artifice is devised whereby it may also be made to furnish the equations for the other elements. It would only need a slight amount of modification to obtain the equations for all the elements simultaneously by straightforward analysis.

This paper also contains an investigation of the motion of a satellite moving about an oblate planet by means of equations, which give simultaneously the nutations of the planet and the corresponding inequalities in the motion of the satellite. The equations are afterwards extended so as to include the effects of tidal friction. I found this portion of the work far more arduous than anything else in the whole series of researches.

The developments and integrations in all these papers are carried out with what may perhaps be regarded as an unnecessary degree of elaboration, but it was impossible to foresee what terms might become important. It does not, however, seem worth while to comment further on minor points such as these.
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PREFACE.

For the astronomer who is interested in cosmogony the important point is the degree of applicability of the theory as a whole to celestial evolution. To me it seems that the theory has rather gained than lost in the esteem of men of science during the last 25 years, and I observe that several writers are disposed to accept it as an established acquisition to our knowledge of cosmogony.

Undue weight has sometimes been laid on the exact numerical values assigned for defining the primitive configuration of the earth and moon. In so speculative a matter close accuracy is unattainable, for a different theory of frictionally retarded tides would inevitably lead to a slight difference in the conclusion; moreover such a real cause as the secular increase in the masses of the earth and moon through the accumulation of meteoric dust, and possibly other causes are left out of consideration.

The exact nature of the process by which the moon was detached from the earth must remain even more speculative. I suggested that the fission of the primitive planet may have been brought about by the synchronism of the solar tide with the period of the fundamental free oscillation of the planet, and the suggestion has received a degree of attention which I never anticipated. It may be that we shall never attain to a higher degree of certainty in these obscure questions than we now possess, but I would maintain that we may now hold with confidence that the moon originated by a process of fission from the primitive planet, that at first she revolved in an orbit close to the present surface of the earth, and that tidal friction has been the principal agent which transformed the system to its present configuration.

The theory for a long time seemed to lie open to attack on the ground that it made too great demands on time, and this has always appeared to me the greatest difficulty in the way of its acceptance. If we were still compelled to assent to the justice of Lord Kelvin's views as to the period of time which has elapsed since the earth solidified, and as to the age of the solar system, we should also have to admit that the theory of evolution under tidal influence is inapplicable to its full extent. Lord Kelvin's contributions to cosmogony have been of the first order of importance, but his arguments on these points no longer carry conviction with them. Lord Kelvin contended that the actual distribution of land and sea proves that the planet solidified at a time when the day had nearly its present length. If this were true the effects of tidal friction relate to a period antecedent to the solidification. But I have always felt convinced that the earth would adjust its ellipticity to its existing speed of rotation with close approximation. The calculations contained in Paper 9, the plasticity of even the most refractory
PREFACE.

forms of matter under great stresses, and the contortions of geological strata appear to me, at least, conclusive against Lord Kelvin's view.

The researches of Mr Strutt on the radio-activity of rocks prove that we cannot regard the earth simply as a cooling globe, and therefore Lord Kelvin's argument as to the age of the earth as derived from the observed gradient of temperature must be illusory. Indeed even without regard to the initial temperature of the earth acquired by means of secular contraction, it is hard to understand why the earth is not hotter inside than it is.

It seems probable that Mr Strutt may be able to obtain a rough numerical scale of geological time by means of his measurements of the radio-activity of rocks, and although he has not yet been able to formulate such a scale with any degree of accuracy, he is already confident that the periods involved must be measured in hundreds or perhaps even thousands of millions of years*. The evidence, taken at its lowest, points to a period many times as great as was admitted by Lord Kelvin for the whole history of the solar system.

Lastly the recent discovery of the colossal internal energy resident in the atom shows that it is unsafe to calculate the age of the sun merely from mechanical energy, as did Helmholtz and Kelvin. It is true that the time has not yet arrived at which we can explain exactly the manner in which the atomic energy may be available for maintaining the sun's heat, but when the great age of the earth is firmly established the insufficiency of the supply of heat to the sun by means of purely mechanical energy will prove that atomic energy does become available in some way. On the whole then it may be maintained that deficiency of time does not, according to our present state of knowledge, form a bar to the full acceptability of the theory of terrestrial evolution under the influence of tidal friction.

It is very improbable that tidal friction has been the dominant cause of change in any of the other planetary sub-systems or in the solar system itself, yet it seems to throw light on the distribution of the satellites amongst the several planets. It explains the identity of the rotation of the moon with her orbital motion, as was long ago pointed out by Kant and Laplace, and it tends to confirm the correctness of the observations according to which Venus always presents the same face to the sun. Finally it has been held by Dr See and by others to explain some of the peculiarities of the orbits of double stars.

Lord Kelvin's determination of the strain of an elastic sphere and the solution of the corresponding problem of the tides of a viscous spheroid suggested another interesting question with respect to the earth. This problem is to find the strength of the materials of which the earth must be

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built so as to prevent the continents from sinking and the sea bed from rising; this question is treated in Paper 9 (p. 459). The existence of an isostatic layer, at which the hydrostatic pressure is uniform, at no great depth below the earth's surface, is now well established. This proves that I have underestimated in my paper the strength of the superficial layers necessary to prevent subsidence and elevation. The strength of granite and of other rocks is certainly barely adequate to sustain the continents in position, and Mr Hayford* seeks to avoid the difficulty by arguing that the earth is actually ‘a failing structure,’ and that the subsidence of the continents is only prevented by the countervailing effects of the gradually increasing weight of sedimentation on the adjoining sea-beds.

In his address to the Geological Section of the British Association at Dublin (1908) Professor Joly makes an interesting suggestion which bears on this subject. He supposes that the heat generated by the radio-active materials in sediment has exercised an important influence in bringing about the elevation of mountain ranges and of the adjoining continents.

A subsidiary outcome of this same investigation was given in Vol. 1. of these papers, when I attempted to determine the elastic oscillations of the superficial layers of the earth under the varying pressures of the tides and of the atmosphere. Dr Hecker may perhaps be able to verify or disprove these theoretical calculations when he makes the final reduction of his valuable observations with horizontal pendulums at Potsdam.

When the first volume of these papers was published Lord Kelvin was still alive, and I had the pleasure of receiving from him a cordial letter of thanks for my acknowledgement of the deep debt I owe him. His name also occurs frequently in the present volume, and if I dissent from some of his views, I none the less regard him as amongst the greatest of those who have tried to guess the riddle of the history of the universe.

The chronological list of my papers is repeated in this second volume, together with a column showing in which volume they are or will be reproduced.

In conclusion I wish to thank the printers and readers of the Cambridge University Press for their marvellous accuracy and care in setting up the type and in detecting some mistakes in the complicated analysis contained in these papers.

G. H. DARWIN.

October, 1908.

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2. Note on Thomson's Theory of the Tides of an Elastic Sphere  
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Unpublished Article ‘Tides.’ Encyclopaedia Britannica, new edition to be published hereafter (by permission of the proprietors).

Certain sections in I

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### ERRATUM IN VOL. I.

p. 275, equation (26), line 9 from foot of page, should read

\[ 3a = 4\nu + 2\mu \kappa_m - e^2. \]