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On the local structure of ordinary Hecke algebras at classical weight one points

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Abstract

The aim of this chapter is to explain how one can obtain information regarding the membership of a classical weight one eigenform in a Hida family from the geometry of the Eigencurve at the corresponding point. We show, in passing, that all classical members of a Hida family, including those of weight one, share the same local type at all primes dividing the level.

1. Introduction

Classical weight one eigenforms occupy a special place in the correspondence between Automorphic Forms and Galois Representations since they yield two dimensional Artin representations with odd determinant. The construction of those representations by Deligne and Serre [5] uses congruences with modular forms of higher weight. The systematic study of congruences between modular forms has culminated in the construction of the p-adic Eigencurve by Coleman and Mazur [4]. A p-stabilized classical weight one eigenform corresponds then to a point on the ordinary component of the Eigencurve, which is closely related to Hida theory.

An important result of Hida [11] states that an ordinary cuspform of weight at least two is a specialization of a unique, up to Galois conjugacy, primitive Hida family. Geometrically this translates into the smoothness of the Eigencurve at that point (in fact, Hida proves more, namely that the map

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to the weight space is etale at that point). Whereas Hida's result continues to hold at all non-critical classical points of weight two or more [13], there are examples where this fails in weight one [6]. The purely quantitive question of how many Hida families specialize to a given classical p-stabilized weight one eigenform, can be reformulated geometrically as to describe the local structure of the Eigencurve at the corresponding point. An advantage of the new formulation is that it provides group theoretic and homological tools for the study of the original question thanks to Mazur's theory of deformations of Galois representations. Moreover, this method gives more qualitative answers, since the local structure of the Eigencurve at a given point contains more information than the collection of all Hida families passing through that point.

The local structure at weight one forms with RM was first investigated by Cho and Vatsal [3] in the context of studying universal deformation rings, who showed that in many cases the Eigencurve is smooth, but not etale over the weight space, at those points. The main result of a joint work with Joël Bellaïche [1] states that the p-adic Eigencurve is smooth at all classical weight one points which are regular at p and gives a precise criterion for etalness over the weight space at those points. The author has learned recently that the work [10] of Greenberg and Vatsal contains a slightly weaker version of this result. It would be interesting to describe the local structure at irregular points, to which we hope to come back in a future work.

The chapter is organized as follows. Section 2 describes some p-adic aspects in the theory of weight one eigenforms. Sections 3 and 4 introduce, respectively, the ordinary Hecke algebras and primitive Hida families, which are central objects in Hida theory [12]. In Section 5 various Galois representations are studied with emphasis on stable lattices, leading to the construction of a representation (1.10) which is a bridge between a primitive Hida family and its classical members. This is used in Section 6 to establish the rigidity of the local type in a Hida family, including in weight one (see Proposition 1.8). Section 7 quotes the main results of [1] and describes their consequences in classical Hida theory (see Corollary 1.15). The latter would have been rather straightforward, should the Eigencurve have been primitive, in the sense that the irreducible component of its ordinary locus would have corresponded (after inverting p) to primitive Hida families. Lacking a reference for the construction of such an Eigencurve, we establish a local isomorphism, at the points of interest, between the reduced Hecke algebra, used in the definition of the Eigencurve, and the new quotient of the full Hecke algebra, used in the definition of primitive Hida families (see Corollary 1.14).

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2. Artin modular forms and the Eigencurve

We let $\bar{\mathbb{Q}} \subset \mathbb{C}$ be the field of algebraic numbers, and denote by $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ the absolute Galois group of $\mathbb{Q}$. For a prime ℓ we fix a decomposition subgroup G_ℓ of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and denote by I_ℓ its inertia subgroup and by Frob_ℓ the arithmetic Frobenius in G_ℓ/I_ℓ.

We fix a prime number p and an embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$.

Let $f(z) = \sum_{n\geq 1} a_n q^n$ be a newform of weight one, level M and central character ϵ. Thus $a_1 = 1$ and for every prime $\ell \nmid M$ (resp. $\ell \mid M$) f is an eigenvector with eigenvalue a_ℓ for the Hecke operator T_ℓ (resp. U_ℓ). By a theorem of Deligne and Serre [5] there exists a unique continuous irreducible representation:

$$\rho_f : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C}), \tag{1.1}$$

such that its Artin L-function $L(\rho_f, s)$ equals

$$L(f,s) = \sum_n \frac{a_n}{n^s} = \prod_{\ell\nmid M}(1 - a_\ell \ell^{-s} + \epsilon(\ell)\ell^{-2s})^{-1} \prod_{\ell\mid M}(1 - a_\ell \ell^{-s})^{-1}.$$

It follows that if $a_\ell \neq 0$ for $\ell \mid M$, then a_ℓ is the eigenvalue of $\rho_f(\mathrm{Frob}_\ell)$ acting on the unique line fixed by I_ℓ. Since ρ_f has finite image, a_ℓ is an eigenvalue of a finite order matrix, hence it is a root of unity.

Similarly, for $\ell \nmid M$ the characteristic polynomial $X^2 - a_\ell X + \epsilon(\ell)$ of $\rho_f(\mathrm{Frob}_\ell)$ has two (possibly equal) roots α_ℓ and β_ℓ which are both roots of unity.

In order to deform f p-adically, one should first choose a p-stabilization of f with finite slope, that is an eigenform of level $\Gamma_1(M) \cap \Gamma_0(p)$ sharing the same eigenvalues as f away from p and having a non-zero U_p-eigenvalue. By the above discussion if such a stabilization exists, then it should necessarily be ordinary. We distinguish two cases:

If p does not divide M, then f has two p-stabilizations $f_\alpha(z) = f(z) - \beta_p f(pz)$ and $f_\beta(z) = f(z) - \alpha_p f(pz)$ with U_p-eigenvalue α_p and β_p, respectively.

If p divides M and $a_p \neq 0$, then f is already p-stabilized. We let then $\alpha_p = a_p$ and $f_\alpha = f$.

Denote by N the prime to p-part of M.

Definition 1.1. We say that f_α is *regular* at p if either p divides M and $a_p \neq 0$, or p does not divide M and $\alpha_p \neq \beta_p$.

The Eigencurve $\mathcal{C}$ of tame level $\Gamma_1(N)$ is a rigid analytic curve over $\mathbb{Q}_p$ parametrizing systems of eigenvalues for the Hecke operators T_ℓ ($\ell \nmid Np$) and U_p appearing in the space of finite slope overconvergent modular forms of tame level dividing N. We refer to the original article of Coleman and Mazur [4] for the case $N = 1$ and $p > 2$, and to Buzzard [2] for the general case. Recall that $\mathcal{C}$ is reduced and endowed with a flat and locally finite weight map $\kappa : \mathcal{C} \to \mathcal{W}$, where $\mathcal{W}$ is the rigid space over $\mathbb{Q}_p$ representing homomorphisms $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{G}_m$.

The p-stabilized newform f_α defines a point on the ordinary component of $\mathcal{C}$, whose image by κ is a character of finite order.

Theorem 1.2. [1] *Let f be a classical weight one cuspidal eigenform form which is regular at p. Then the Eigencurve $\mathcal{C}$ is smooth at the point defined by f_α, so there is a unique irreducible component of $\mathcal{C}$ containing that point. In particular, if f has CM by a quadratic field in which p splits, then all classical points of that component also have CM by the same field.*

Moreover, $\mathcal{C}$ is etale over the weight space $\mathcal{W}$ at the point defined by f_α, unless f has RM by a quadratic field in which p splits.

In Section 7 we will revisit this theorem from the perspective of Hida families.

3. Ordinary Hecke algebras

The results in this and the following two sections are due to Hida [11, 12] when p is odd and have been completed for $p = 2$ by Wiles [18] and Ghate–Kumar [8].

Let $\Lambda = \mathbb{Z}_p[[\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]] \simeq \mathbb{Z}_p[[1 + p^\nu\mathbb{Z}_p]]$ be the Iwasawa algebra of the cyclotomic $\mathbb{Z}_p$ extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$, where $\nu = 2$ if $p = 2$ and $\nu = 1$ otherwise. It is a complete local $\mathbb{Z}_p$-algebra which is an integral domain of Krull dimension 2. Let χ_{cyc} be the universal Λ-adic cyclotomic character obtained by composing $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ with the canonical continuous group homomorphism from $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ to the units of its completed group ring Λ.

We say that a height one prime ideal $\mathfrak{p}$ of a finite Λ-algebra $\mathbb{T}$ is of weight k (an integer ≥ 1) if $P = \mathfrak{p} \cap \Lambda$ is the kernel of a $\mathbb{Z}_p$-algebra homomorphism $\Lambda \to \bar{\mathbb{Q}}_p$ whose restriction to a finite index subgroup of $1 + p^\nu\mathbb{Z}_p$ is given by $x \mapsto x^{k-1}$. Such an ideal $\mathfrak{p}$ induces a Galois orbit of $\mathbb{Z}_p$-algebra homomorphisms $\mathbb{T} \to \mathbb{T}/\mathfrak{p} \hookrightarrow \bar{\mathbb{Q}}_p$ called specializations in weight k.

By definition a Λ-adic ordinary cuspform of level N (a positive integer not divisible by p) is a formal q-expansion with coefficients in the integral closure of Λ in some finite extension of its fraction field, whose specialization in any weight $k \geq 2$ yield the q-expansion of a p-stabilized, ordinary, normalized cuspform of tame level N and weight k. However, specializations in weight one are not always classical.

The ordinary Hecke algebra $\mathbb{T}_N$ of tame level N is defined as the Λ-algebra generated by the Hecke operators U_ℓ (resp. T_ℓ, $\langle \ell \rangle$) for primes ℓ dividing Np (resp. not dividing Np) acting on the space of Λ-adic ordinary cuspforms of tame level N. Hida proved that $\mathbb{T}_N$ is free of finite rank over Λ and its height one primes of weight $k \geq 2$ are in bijection with the (Galois orbits of) classical ordinary eigenforms of weight k and tame level dividing N.

A Λ-adic ordinary cuspform of level N is said to be N-new if all specializations in weights ≥ 2 are p-stabilized, ordinary cuspforms of tame level N which are N-new.

Define $\mathbb{T}_N^{\mathrm{new}}$ as the quotient of $\mathbb{T}_N$ acting faithfully on the space of Λ-adic ordinary cuspforms of level N, which are N-new. A result of Hida (see [12, Corollaries 3.3 and 3.7]) states that $\mathbb{T}_N^{\mathrm{new}}$ is a finite, reduced, torsion free Λ-algebra, whose height one primes of weight $k \geq 2$ are in bijection with the Galois orbits of classical ordinary eigenforms of weight k and tame level N which are N-new.

4. Primitive Hida families

A primitive Hida family $F = \sum_{n \geq 1} A_n q^n$ of tame level N is by definition a Λ-adic ordinary cuspform, new of level N and which is a normalized eigenform for all the Hecke operators, i.e., a common eigenvector of the operators U_ℓ, T_ℓ and $\langle \ell \rangle$ as above. The relations between coefficients and eigenvalues for the Hecke operators are the usual ones for newforms. One can see from [12, p. 265] that primitive Hida families can be used to write down a basis of the space of Λ-adic ordinary cuspforms in the same fashion as classically newforms can be used to write down a basis of the space of cuspforms.

The central character $\psi_F : (\mathbb{Z}/Np^\nu)^\times \to \mathbb{C}^\times$ of the family is defined by $\psi_F(\ell) =$ eigenvalue of $\langle \ell \rangle$.

Galois orbits of primitive Hida families of level N are in bijection with the minimal primes of $\mathbb{T} = \mathbb{T}_N^{\mathrm{new}}$. More precisely, a primitive Hida family determines and is uniquely determined by a Λ-algebra homomorphism $\mathbb{T} \to \overline{\mathrm{Frac}(\Lambda)}$, sending each Hecke operator to its eigenvalue on F, whose kernel is a minimal prime $\mathfrak{a} \subset \mathbb{T}$. Since $\mathbb{T}$ is a finite and reduced Λ-algebra, its localization $\mathbb{T}_\mathfrak{a}$ is a finite field extension of $\mathrm{Frac}(\Lambda)$. Hence, we obtain the following homomorphisms of Λ-algebras:

$$\mathbb{T} \twoheadrightarrow \mathbb{T}/\mathfrak{a} \hookrightarrow \widetilde{\mathbb{T}/\mathfrak{a}} \hookrightarrow \mathbb{T}_\mathfrak{a} \xrightarrow{\sim} K_F \subset \overline{\mathrm{Frac}(\Lambda)}, \tag{1.2}$$

where $\widetilde{\mathbb{T}/\mathfrak{a}}$ denotes the integral closure of the domain $\mathbb{T}/\mathfrak{a}$ in its field of fractions $\mathbb{T}_{\mathfrak{a}}$. In particular, the image K_F of $\mathbb{T}_{\mathfrak{a}}$ in $\overline{\mathrm{Frac}(\Lambda)}$ is a finite extension of $\mathrm{Frac}(\Lambda)$ generated by the coefficients of F.

By definition all specializations of F in weight $k \geq 2$ yield p-stabilized, ordinary newforms of tame level N and weight k. In weight one, there are only finitely many classical specializations, unless F has CM by a quadratic field in which p splits (see [9] and [6]). Nevertheless, a theorem of Wiles [18] asserts that any p-stabilized newform of weight one occurs as a specialization of a primitive Hida family.

Given a primitive Hida family $F = \sum_{n\geq 1} A_n q^n$ of level N, Hida constructed in [11, Theorem 2.1] an absolutely irreducible continuous representation:

$$\rho_F : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(K_F), \tag{1.3}$$

unramified outside Np, such that for all ℓ not dividing Np the trace of the image of Frob_ℓ equals A_ℓ. Moreover $\det \rho_F = \psi_F \chi_{\mathrm{cyc}}$. Finally by Wiles [18, Theorem 2.2.2] the space of I_p-coinvariants is a line on which Frob_p acts by A_p.

5. Galois representations

5.1. Minimal primes

The total quotient field of $\mathbb{T}$ is given by $\mathbb{T} \otimes_\Lambda \mathrm{Frac}(\Lambda) \simeq \prod_{\mathfrak{a}} \mathbb{T}_{\mathfrak{a}}$ where the product is taken over all minimal primes of $\mathbb{T}$. The representation (1.3) can be rewritten as

$$\rho_{\mathfrak{a}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{T}_{\mathfrak{a}}) \tag{1.4}$$

and by putting those together we obtain a continuous representation

$$\rho_{\mathbb{T}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{T} \otimes_\Lambda \mathrm{Frac}(\Lambda)) \tag{1.5}$$

unramified outside Np, such that for all ℓ not dividing Np the trace of the image of Frob_ℓ equals T_ℓ. Moreover the space of I_p-coinvariants is free of rank one and Frob_p acts on it as U_p.

5.2. Maximal primes

Since $\mathbb{T}$ is a finite Λ-algebra, it is semi-local, and is isomorphic to the direct product $\prod_{\mathfrak{m}} \mathbb{T}_{\mathfrak{m}}$ where the product is taken over all maximal primes. By composing (1.5) with the canonical projection, one obtains:

$$\rho_{\mathfrak{m}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{T}_{\mathfrak{m}} \otimes_\Lambda \mathrm{Frac}(\Lambda)). \tag{1.6}$$

The composition:

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\mathrm{Tr}(\rho_{\mathfrak{m}})} \mathbb{T}_{\mathfrak{m}} \to \mathbb{T}/\mathfrak{m} \tag{1.7}$$

is a pseudo-character taking values in a field and sending the complex conjugation to 0. By a result of Wiles [18, §2.2] it is the trace of a unique semi-simple representation:

$$\bar{\rho}_{\mathfrak{m}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{T}/\mathfrak{m}). \tag{1.8}$$

Note that whereas each minimal prime $\mathfrak{a} \subset \mathbb{T}$ is contained in a unique maximal prime, there may be several minimal primes contained in a given maximal prime $\mathfrak{m}$, those corresponding to primitive Hida families sharing the same residual Galois representation $\bar{\rho}_{\mathfrak{m}}$.

5.3. Galois stable lattices

A lattice over a noetherian domain R (or R-lattice) is a finitely generated R-submodule of a finite dimensional $\mathrm{Frac}(R)$-vector space which spans the latter. This definition extends to a noetherian reduced ring R and its total quotient field $\prod_{\mathfrak{a}} R_{\mathfrak{a}}$, where $\mathfrak{a}$ runs over the (finitely many) minimal primes of R.

The continuity of $\rho_{\mathfrak{a}}$ implies the existence of a Galois stable $\widetilde{\mathbb{T}/\mathfrak{a}}$-lattice in $\mathbb{T}_{\mathfrak{a}}^2$, and similar statements hold for ρ_F, $\rho_{\mathbb{T}}$ and $\rho_{\mathfrak{m}}$. It is worth mentioning that $\rho_{\mathbb{T}}$ cannot necessarily be defined over the normalization of $\mathbb{T}$ in $\prod_{\mathfrak{a}} \mathbb{T}_{\mathfrak{a}}$. In other words $\rho_{\mathfrak{a}}$ does not necessarily stabilize a *free* $\widetilde{\mathbb{T}/\mathfrak{a}}$-lattice. There is an exception: if $K_F = \mathrm{Frac}(\Lambda)$ and $p > 2$ the regularity of Λ implies that ρ_F always admits a Galois stable free Λ-lattice (see [11, §2]).

If $\mathfrak{m}$ is a maximal prime such that the residual Galois representation $\bar{\rho}_{\mathfrak{m}}$ is absolutely irreducible, then by a result of Nyssen [15] and Rouquier [16] $\rho_{\mathfrak{m}}$ stabilizes a free $\mathbb{T}_{\mathfrak{m}}$-lattice. It follows that for every minimal prime $\mathfrak{a} \subset \mathfrak{m}$, the representation $\rho_{\mathfrak{a}}$ stabilizes a free lattice over $\mathbb{T}_{\mathfrak{m}}/\mathfrak{a} = \mathbb{T}/\mathfrak{a}$.

5.4. Height one primes

Let f be a p-stabilized, ordinary, newform of tame level N and weight k. It determines uniquely a height one prime $\mathfrak{p} \subset \mathbb{T}$ and an embedding of $\mathbb{T}_{\mathfrak{p}}/\mathfrak{p}$ into $\bar{\mathbb{Q}}_p$, although not every height one prime of $\mathbb{T}$ of weight one is obtained in this way. Our main interest is in the structure of the Λ_P-algebra $\mathbb{T}_{\mathfrak{p}}$, where $P = \mathfrak{p} \cap \Lambda$. The ring $\mathbb{T}_{\mathfrak{p}}$ is local, noetherian, reduced of Krull dimension 1, but is not necessarily integrally closed. It might even not be a domain, since f

could be a specialization of several, non Galois conjugate, Hida families (see [6, §7.4]), hence there may be several minimal primes $\mathfrak{a}$ of $\mathbb{T}$ contained in $\mathfrak{p}$.

Let $\rho_f : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ be the continuous irreducible representation attached to f by Deligne when $k \geq 2$ and by (1.1) when $k = 1$ via the fixed embeddings $\mathbb{C} \supset \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Since ρ_f is odd, it can be defined over the ring of integers of the subfield of $\bar{\mathbb{Q}}_p$ generated by its coefficients, and hence defines an isomorphic representation:

$$\bar{\rho}_{\mathfrak{p}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{T}_{\mathfrak{p}}/\mathfrak{p}), \tag{1.9}$$

admitting a model over the integral closure of $\mathbb{T}/\mathfrak{p}$ in its field of fractions $\mathbb{T}_{\mathfrak{p}}/\mathfrak{p}$.

The normalization of $\mathbb{T}_{\mathfrak{p}}$ in its total quotient field $\prod_{\mathfrak{a}\subset\mathfrak{p}} \mathbb{T}_{\mathfrak{a}}$ is given by $\prod_{\mathfrak{a}\subset\mathfrak{p}} \widetilde{\mathbb{T}_{\mathfrak{p}}/\mathfrak{a}}$, where $\widetilde{\mathbb{T}_{\mathfrak{p}}/\mathfrak{a}} \simeq \widetilde{(\mathbb{T}/\mathfrak{a})}_{\mathfrak{p}}$ is the integral closure of $\mathbb{T}_{\mathfrak{p}}/\mathfrak{a} \simeq (\mathbb{T}/\mathfrak{a})_{\mathfrak{p}}$ in $\mathbb{T}_{\mathfrak{a}}$.

Denote by $\widehat{\mathbb{T}_{\mathfrak{p}}/\mathfrak{a}}$ the completion of the discrete valuation ring $\widetilde{\mathbb{T}_{\mathfrak{p}}/\mathfrak{a}}$. Note that they share the same residue field which is a finite extension of $\mathbb{T}_{\mathfrak{p}}/\mathfrak{p}$ and that there is a natural bijection between the set of $\widetilde{\mathbb{T}_{\mathfrak{p}}/\mathfrak{a}}$-lattices in a given $\mathbb{T}_{\mathfrak{a}}$-vector space V and the set of $\widehat{\mathbb{T}_{\mathfrak{p}}/\mathfrak{a}}$-lattices in $V \otimes_{\mathbb{T}_{\mathfrak{a}}} \mathrm{Frac}(\widehat{\mathbb{T}_{\mathfrak{p}}/\mathfrak{a}})$. Since $\bar{\rho}_{\mathfrak{p}}$ is absolutely irreducible and $\widehat{\mathbb{T}_{\mathfrak{p}}/\mathfrak{a}}$ is local and complete, by a result of Nyssen [15] and Rouquier [16] the representation $\rho_{\mathfrak{a}} \otimes_{\mathbb{T}_{\mathfrak{a}}} \mathrm{Frac}(\widehat{\mathbb{T}_{\mathfrak{p}}/\mathfrak{a}})$ stabilizes a free $\widehat{\mathbb{T}_{\mathfrak{p}}/\mathfrak{a}}$-lattice. The latter lattice yields (by intersection) a free $\widetilde{\mathbb{T}_{\mathfrak{p}}/\mathfrak{a}}$-lattice stable by $\rho_{\mathfrak{a}}$. In other terms there exists a unique, up to conjugacy, continuous representation:

$$\rho_{\mathfrak{p}}^{\mathfrak{a}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\widetilde{\mathbb{T}_{\mathfrak{p}}/\mathfrak{a}}), \tag{1.10}$$

such that $\rho_{\mathfrak{p}}^{\mathfrak{a}} \otimes_{\widetilde{\mathbb{T}_{\mathfrak{p}}/\mathfrak{a}}} \mathbb{T}_{\mathfrak{a}} \simeq \rho_{\mathfrak{a}}$ and $\rho_{\mathfrak{p}}^{\mathfrak{a}} \mod \mathfrak{p} \simeq \bar{\rho}_{\mathfrak{p}}$.

This representation is a bridge between a form and a family and will be used in Section 6 to transfer properties in both directions.

The exact control theorem for ordinary Hecke algebras, proved by Hida for $p > 2$ and by Ghate–Kumar [8] for $p = 2$, has the following consequence:

Theorem 1.3. [11, Corollary 1.4] *Assume that $k \geq 2$. Then the local algebra $\mathbb{T}_{\mathfrak{p}}$ is etale over the discrete valuation ring Λ_P. In particular, f is a specialization of a unique, up to Galois conjugacy, Hida family corresponding to a minimal prime $\mathfrak{a}$.*

Assume for the rest of this section that $\mathbb{T}_{\mathfrak{p}}$ is a domain. Then the field of fractions of $\mathbb{T}_{\mathfrak{p}}$ is isomorphic to $\mathbb{T}_{\mathfrak{a}}$, where $\mathfrak{a}$ is the unique minimal prime of $\mathbb{T}$ contained in $\mathfrak{p}$. Since normalization and localization commute, we have

$\widetilde{(\mathbb{T}/\mathfrak{a})_{\mathfrak{p}}} \simeq (\widetilde{\mathbb{T}/\mathfrak{a}})_{\mathfrak{p}} \simeq \widetilde{\mathbb{T}_{\mathfrak{p}}}$. Therefore, the collection of representations (1.10) are replaced by a unique, up to conjugacy, continuous representation:

$$\rho_{\mathfrak{p}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\widetilde{\mathbb{T}_{\mathfrak{p}}}), \tag{1.11}$$

such that $\rho_{\mathfrak{p}} \otimes_{\widetilde{\mathbb{T}_{\mathfrak{p}}}} \mathbb{T}_{\mathfrak{a}} \simeq \rho_{\mathfrak{a}}$ and $\rho_{\mathfrak{p}} \mod \mathfrak{p} \simeq \bar{\rho}_{\mathfrak{p}}$.

If we further assume that $\mathbb{T}_{\mathfrak{p}}$ is etale over Λ_P, then $\mathbb{T}_{\mathfrak{p}}$ is itself a discrete valuation ring, hence $\mathbb{T}_{\mathfrak{p}} \simeq \widetilde{\mathbb{T}_{\mathfrak{p}}}$.

6. Rigidity of the automorphic type in a Hida family

By definition, all specializations in weight at least two of a primitive Hida family F of level N share the same tame level. Also, by [7, Proposition 2.2.4], the tame conductor of ρ_F equals N. The aim of this section is to show that the tame level of all classical weight one specializations of F is also N, and to show that all classical specializations of F (including those of weight one) share the same automorphic type at all primes dividing N.

6.1. Minimally ramified Hida families

Recall that a newform f is said to be minimally ramified if it has minimal level amongst the underlying newforms of all its twists by Dirichlet characters.

Lemma 1.4. *Let F be a primitive Hida family and let χ be a Dirichlet character of conductor prime to p. There exists a unique primitive Hida family F_χ underlying $F \otimes \chi$, in the sense that the p-stabilized, ordinary newform underlying a given specialization of $F \otimes \chi$ can be obtained by specializing F_χ.*

Proof. By [12, p. 250] one can write any Λ-adic ordinary cuspform as a linear combination of translates of primitive Hida families of lower or equal level. Since $F \otimes \chi$ is an eigenform for all but finitely many Hecke operators, it is necessarily a linear combination of translates of the same primitive Hida family, denoted F_χ. It follows that any specialization of F_χ in weight at least two is the p-stabilized, ordinary newform underlying the corresponding specialization of $F \otimes \chi$. □

Definition 1.5. We say that a primitive Hida family F of level N is minimally ramified if for every Dirichlet character χ of conductor prime to p, the level of F_χ is a multiple of N.

As for newforms, it is clear that any primitive Hida family admits a unique twist which is minimally ramified.

Lemma 1.4 implies that being minimally ramified is pure with respect to specializations in weight at least two, that is to say, all specializations of a minimally ramified primitive Hida family are minimally ramified, and a primitive Hida family admitting a minimally ramified specialization is minimally ramified. This observation together with the classification of the admissible representations of $\mathrm{GL}_2(\mathbb{Q}_\ell)$, easily implies:

Lemma 1.6. *Let $F = \sum_{n\geq 1} A_n q^n$ be a minimally ramified, primitive Hida family of level N and let ℓ be a prime dividing N. Denote by $\mathrm{unr}(C)$ the unramified character of G_ℓ sending Frob_ℓ to C.*

(i) *If ψ_F is unramified at ℓ and ℓ^2 does not divide N, then every specialization in weight at least two corresponds to an automorphic form which is special at ℓ. In particular $A_\ell \neq 0$ and the restriction of ρ_F to G_ℓ is an unramified twist of an extension of 1 by $\mathrm{unr}(\ell)$.*
(ii) *If the conductor of ψ_F and N share the same ℓ-part, then every specialization in weight at least two corresponds to an automorphic form which is a ramified principal series at ℓ. In particular $A_\ell \neq 0$ and the restriction of ρ_F to G_ℓ equals $\mathrm{unr}(A_\ell) \oplus \mathrm{unr}(B_\ell)\psi_F$, for some $B_\ell \in K_F$.*
(iii) *In all other cases, every specialization in weight at least two corresponds to an automorphic form which is supercuspidal at ℓ. In particular $A_\ell = 0$ and the restriction of ρ_F to G_ℓ is irreducible.*

6.2. General case

Definition 1.7. Let F be a primitive Hida family of level N and let ℓ be a prime dividing N. We say that F is special (resp. ramified principal series or supercuspidal) at ℓ, if a minimally ramified twist of F falls in case (i) (resp. (ii) or (iii)) of Lemma 1.6.

It follows from Lemma 1.6, that being special, principal series or supercuspidal is pure with respect to specializations, that is to say, all specializations in weight at least two are of the same type. We will now describe the local automorphy type in greater detail and deduce information about classical weight one specializations.

Proposition 1.8. *Let F be a primitive Hida family of level N and let ℓ be a prime dividing N. If F is special at ℓ, so are all its specializations in weight at least two and F does not admit any classical weight one specialization. Otherwise, $\rho_F(I_\ell)$ is a finite group invariant under any classical specialization, including in weight one. More precisely*