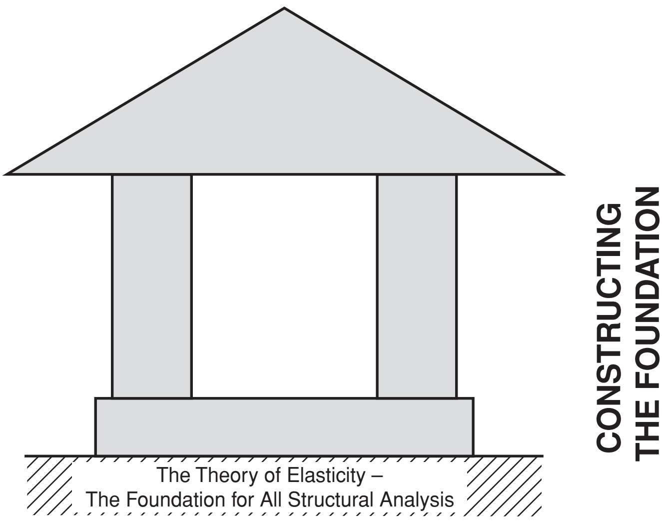
PART I

THE FUNDAMENTALS OF STRUCTURAL ANALYSIS



I.1 An Overview of Part I

Vehicular weight, particularly that of aircraft and spacecraft, has a strong effect on the performance or economics of all such vehicles. Thus it is well worth spending many engineering man-hours on their design and analysis so as to make those vehicles as light-weight as possible. To make those many engineering hours of analysis as effective as possible, it is important that all the different types of loads that the vehicle will bear be well estimated, and then the structural response to those loads be carefully calculated. To carefully calculate the response of structures to estimated or measured loadings, it is important to use structural analysis techniques to which considerable confidence can be assigned. High degrees of confidence are achieved through experience and through thorough understanding of any approximations that are incorporated within the derivations of the selected structural analysis techniques. Thus it would seem that, in general, the fewer and the smaller the approximations, the more useful the structural analysis technique. This surmise is only partially true. As will be seen as the material of this textbook unfolds, the use of structural analysis techniques that contain essentially no approximations for many circumstances can be much too expensive and time consuming. Hence a compromise between cost and accuracy is necessary for good engineering practice. To understand how that compromise is found, this introduction to aerospace structures begins with the fundamentals of structural mechanics where the approximations are few in number and small in impact.

Part I of this textbook presents structural mechanics on a differential scale. That is, the focus of the analysis is typically on a volume of engineering material whose rectangular volume is $dx\,dy\,dz$. The enormous advantage of this approach is that the equations that are so established by this process apply to any type of component (beam, shell, solid) of any engineering structure simply because such a differential volume can be visualized as being part of the finite volume of that type of structural component. The frequent use of differential distances like dx should suggest that the calculus, a powerful analytic tool, figures prominently in Part I. The calculus is also vital to the remainder of the textbook because so much of that remainder is based upon the material of Part I. Not only is a knowledge of differential and integral calculus important, but certain other calculus-related aspects of mathematics should be well understood. The remaining four sections of this preface to Part I provide a review of those additional mathematical topics that are essential for a thorough understanding of the Part I material. Knowing the required mathematics makes the engineering much easier.

I.2 Summary of Taylor's Series

Let $f(x)$ be a function of single variable. The Taylor's series for $f(x)$ about $x = a$ may be written as

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \cdots + \frac{1}{n!}(x-a)^n f^{(n)}(a) + \cdots$$

when all the derivatives exist and are continuous in a closed interval containing $x = a$. This same series is written in a slightly different style at the end of this section. The question of exactly when a Taylor's series is valid is not a simple one. The use of Taylor's series to represent the exceptionally smooth functions that generally describe stresses, strains, displacements, and the derivatives of these quantities in continuous structures has never led to contradictions. Hence this series is used freely whenever discontinuities are not suspected. A function that has a Taylor's series expansion is called "analytic."

In two dimensions, at $x = a$ and $y = b$, Taylor's series, written in a slightly different style, is

$$F(a+h, b+k) = F(a,b) + \sum \frac{1}{n!}\left[h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right]^n F(x,y)\Big|_{x=a,\,y=b}$$

where the summation is from $n = 1$ to infinity. If the reader finds the style of presentation for the Taylor's series in two dimensions unfamiliar, it may help to note that, for example, the first series can be written in the style of the second series simply by substituting $(a + h)$ for the variable x. That is, where h is now the variable

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \cdots$$

I.3 Summary of Newton's Method for Finding Roots

There are numerous approximate methods for finding the roots of polynomial equations, many of which are not limited to real roots. Newton's method is a simple matter when limited to real roots, and this method is not limited to polynomial equations. Newton's method is an iterative procedure, which means the same procedure is applied repeatedly until the results exhibit convergence to the degree of accuracy desired. Newton's method

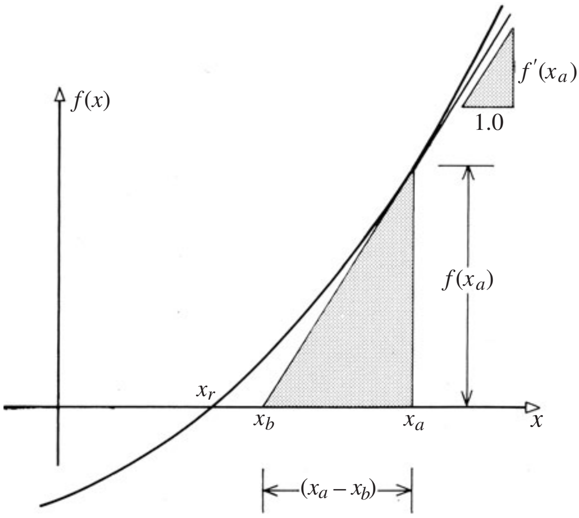


Figure I.1. A tangent (as opposed to secant) approach to determining the real roots of a single variable function.

begins with a first estimate for the location of the desired polynomial root, x_a. It does not matter how this initial estimate x_a is obtained. For example, the initial estimate of the root could be obtained from a rough graph of the polynomial equation. The first estimate is used in this iterative procedure to calculate a second estimate that is closer to the actual root, and the second estimate is used to calculate a still closer third estimate, and so forth. From Fig. I.1, it can be seen that from the interpretation of the derivative as a slope, $f'(x_a) = f(x_a)/(x_a - x_b)$. Solving this equation for the second estimate x_b yields

$$x_b = x_a - \frac{f(x_a)}{f'(x_a)}$$

Used repeatedly, this equation is the means of obtaining a series of improved estimates. The only caution is that the initial estimate has to be "close" enough to the desired root so that the process converges to that root. For example, if it were desired to discover the root $x = \pi$ of the equation $\sin x = 0$, then an initial guess of $x_a = 1$ would lead to the root $x = 0$ rather than the desired root.

See Refs. [44, 60] for a discussion of the intricacies of using Newton's method to find complex roots.

I.4 The Binomial Series

From Ref. [1], it may be proved via use of Taylor's series, that for any real number m, and for any x such that $|x| < 1$,

$$(1+x)^m = 1 + mx + m(m-1)\frac{x^2}{2!} + m(m-1)(m-2)\frac{x^3}{3!} + \cdots + [m(m-1)\cdots(m-n+1)]\frac{x^n}{n!} + \cdots$$

This series is only of finite length when m is equal to a positive integer.

I.5 The Chain Rule for Partial Differentiation

Consider a variable $q = Q(r, s, t)$. In this case, Q is an arbitrary function of the variables r, s, and t, which are called the first class variables. Let the first class variables be in turn functions of the second class variables x, y, and z; that is, $r = R(x, y, z)$, $s = S(x, y, z)$, and $t = T(x, y, z)$. Since the first class variables are dependent on the values of the second class variables, q can also be considered to be a function of the second class variables. Therefore derivatives of q can be taken with respect to the second class variables x, y, and z. The chain rule for partial differentiation of q with respect to x is as follows (Ref. [1]):

$$\frac{\partial q}{\partial x} = \frac{\partial Q}{\partial r}\frac{\partial R}{\partial x} + \frac{\partial Q}{\partial s}\frac{\partial S}{\partial x} + \frac{\partial Q}{\partial t}\frac{\partial T}{\partial x}$$

Notice the pattern of the variables. Each of the first class variables r, s, and t gets its chance to be part of a derivative of Q. Then each first class variable in turn is differentiated with respect to x, which is the second class variable with which the top variable q is differentiated in this illustration. The pattern for $\partial q/\partial x$ is that the leading function of each pair of products is always Q, the trailing variable is always x, and the connecting terms are always related to the first class variables.

The partial derivative of q with respect to y or z is the same as above but for x replaced by y or z. If the first class variables r, s, and t were only functions of x instead of x, y, and z, then $\partial q/\partial x$ would become dq/dx, and $\partial R/\partial x$ would become dR/dx, and so on. It is common practice to write $\partial r/\partial x$ in place of $\partial R/\partial x$, or dr/dx in place of dR/dx, and so on.

CHAPTER 1

Stress in Structures

1.1 The Concept of Stress

Structural engineers are concerned with the effects that forces produce on structures. That forces produce results such as deformations or structural collapse is the usual structural engineering cause-to-effect point of view. Even though this viewpoint is not the only possible or even useful viewpoint, it is the one adopted implicitly in Parts I, II, and III of this text as a temporary convenience until it becomes necessary to adopt a more general viewpoint. In other words, the usual engineering viewpoint is that the forces are an input, the structure is the system, and the effects of the forces acting on the structure (deformations, cracking, etc.) are the output. If a structural effect in turn influences the forces acting on the structure, then a feedback loop involving the forces and the structural effect exists. An example of structural feedback is first encountered in Part III of this text in the form of a beam buckling problem.

The theory that is developed in the next four chapters is valid for *any* type of force or combination of forces (within certain limits), and *any* type of structure. The task of classifying types of forces and structures can wait until it becomes necessary. What is necessary now is to begin to discuss the types of effects that forces produce on structures. One effect that forces can produce is *structural failure.* Structural failure is defined simply as occurring whenever a structure no longer can serve its intended use. A structural failure can be the dramatic collapse or rapid chain reaction disintegration of a large, enormously expensive structure (e.g., the Challenger space shuttle), or it can be as trivial as a wire clothes hanger being sufficiently bent out of shape that its usefulness as a clothes hanger is outweighed by the bother of straightening it. Clearly, certain structural failures are acceptable after an appropriate service life, and the service lives and performance of some structures have to be monitored or ended so as to avoid failures. The resulting question that structural engineers must address is the one that asks how structural failures can be anticipated with reasonable precision; that is, how can failures be predicted mathematically? In order to focus on the preliminary steps essential to predicting structural failures, this text omits discussion of the important topics of confirming mathematical predictions through testing or service experience monitoring.

The question of how to predict structural failures is a difficult question because there are many types of structural failure, and each type of failure has its own complexity. Returning to the example of wire clothes hangers, the large-deformation "bent-out-of-shape" type failure of the wire hanger to support three or more heavy winter coats is quite different from the fracture type of failure that results when a small portion of the hanger is repeatedly bent back and forth upon itself until the wire fractures. The process of predicting structural failures can be conveniently divided into two steps. The first step is the calculation of either or both of the analytical quantities called "stresses" and "displacements." (Definitions of stress and displacement are decided upon later.) The second step is to use, for example, the calculated stresses, the known material characteristics of the structure, and the loading characteristics to estimate the safety of the structure. This introductory text concentrates on

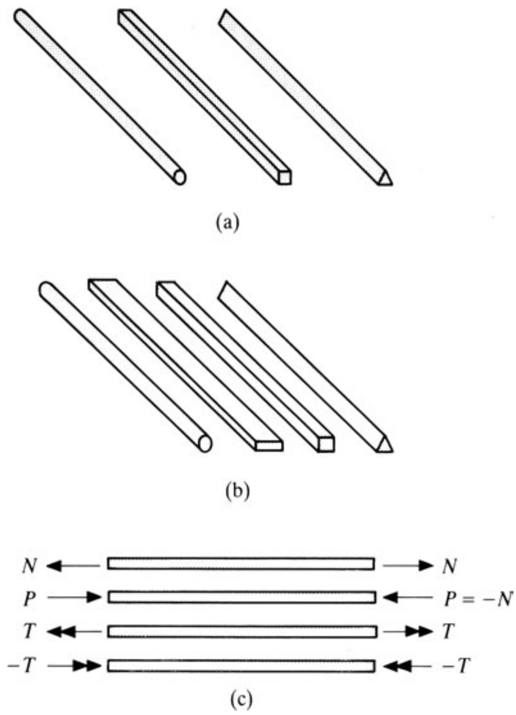


Figure 1.1. (a) Same length, uniform bars with the same cross-sectional area, but different cross-sectional shapes. (b) Same length, uniform bars with twice the cross-sectional area of the previous set. (c) By definition, "bars" only transmit axial forces (tensile or compressive) and twisting moments.

explaining the preliminary step of calculating stresses and displacements. Explanations of the process of using the calculated stresses or displacements to estimate the likelihood of failure is mostly left to more advanced texts, each of which generally concentrates on only one type of failure.

In this chapter the topic of stresses is introduced. The introduction is done in a complete manner that will not require extension or further elaboration short of the most advanced studies in solid mechanics. Thus this approach will save the reader time and effort in the process of learning the elements of structural mechanics. The first thing to be done is to provide an illustration of why engineers have developed the concept of stress, and the usefulness of that concept for determining when a structure will fall in a simple way. The same illustration will provide a basis for choosing a definition for stress. Consider the two sets of bars shown in Figs. 1.1(a) and 1.1(b). A bar is a long thin object of any constant cross-sectional shape that is subjected to only two types of loads. The first type of load is an axial force, that is, a force whose vector representation parallels an axis along the length of the bar. The second type of load is a twisting moment, also called a torque. Its double arrowhead vector representation (right-hand rule) is also one where the vector is parallel to an axis along the length of the bar. The conventional representations of bars loaded in the above manner are shown in Fig. 1.1(c). Let the bars in Fig. 1.1(a) all be well made from the same material and have the same cross-sectional area as that of a typical pencil. Let the bars in Fig. 1.1(b) have twice that cross-sectional area, and be well made of the same material as the bars in Fig. 1.1(a). If increasing tensile forces, that is, forces that tend to stretch the bars, are applied to each of the bars in Fig. 1.1(a), then it would be determined experimentally that all of the bars in Fig. 1.1(a) pulled apart (failed) at approximately the same final value of the applied-tensile force. The small differences between the magnitudes of the tensile forces at failure for each of the bars in Fig. 1.1(a) would be due to experimental measurement errors and small, unobserved differences between the bars. For cross-sectional areas like those of a typical pencil, or greater, it could also be discovered that the length of the bar has no appreciable effect on the magnitude of the failure load. Thus, from this first set of experimental results it can be concluded that, for this type of loading, the shape of the bar cross-section and its length are immaterial. Let the larger cross-section bars of Fig. 1.1(b) be subjected to the same experimental routine. Again it would be observed that each bar of this set failed at approximately the same final load value. Moreover, it would be observed that the failure loads for the bars of Fig. 1.1(b) are twice those of Fig. 1.1(a). In other words, doubling the cross-sectional area doubles the magnitude of the tensile failure load. Furthermore, this proportionality would continue for all larger and many smaller multiples of the cross-sectional area. (If the bar cross-sections are very small, for example, like those of thin wires, then small manufacturing imperfections may have large effects on the magnitude of the tensile load at failure.) Since a consistent goal of all engineers is to simplify their understanding of physical phenomena wherever possible, it is desirable to seek the best possible way to organize this simple data set. This can be done by noticing that the *one* thing that *all* the bars of Figs. 1.1(a) and 1.1(b) have in common is the ratio of the failure load value to the value of the cross-sectional area. The simple experiment described above suggests that the ratio of force to area is a primary means of predicting the behavior of structures and the materials from which they are made. Experiments with different materials, loadings and structural shapes would show that this conclusion is generally true. Hence a special name is bestowed on the ratio of force to area. The name is, of course, *stress*.

Very few useful structures are as simple as the bars in Fig. 1.1. No loadings are simpler than the tensile forces sketched in Fig. 1.1. The latter statement is based on the implication inherent in the sketch as it is drawn that the stress produced by the normal force, N, is evenly (i.e., "uniformly") distributed over the bar cross-sectional area, A. In other words, for this type of loading, the stress everywhere on the bar cross-section is equal to the average stress. In symbolic form, if σ_{av} is the average stress, then by the ratio concept of the preceding paragraph, $\sigma_{av} = N/A$. The question arises as to whether all stress distributions are necessarily uniform. To answer this question, consider two bars of equal length with equal-area square cross-sections. Let the first bar be made of rubber and let the second bar be made of steel. Let axial tensile forces be applied separately to each bar so as to stretch each bar exactly the same distance. The reader recognizes that a much greater force is required to stretch the steel bar the specified distance than is required to stretch the rubber bar that same distance. Thus, in these circumstances of equal areas, the average stress in the steel bar is much greater than the average stress in the rubber bar. Now consider the situation where the two bars in their equally stretched condition are bonded together to form one stretched bar. Clearly, from the viewpoint of the now single bar, the stress distribution is not uniform since the stress is much higher over the steel portion of the bar than over the rubber portion.

It is also true that a nonuniform stress distribution can exist over a cross-section of a bar made of only one material. Consider two square cross-section bars of cross-sectional area A which are made of the same material. Let the first of these two bars be loaded by a tensile force of magnitude 9000 lb, and the second by a tensile force of 1000 lb, where both forces are uniformly distributed over their respective cross-sections. Let the unloaded length of the first bar be just slightly and sufficiently shorter than that of the second bar so

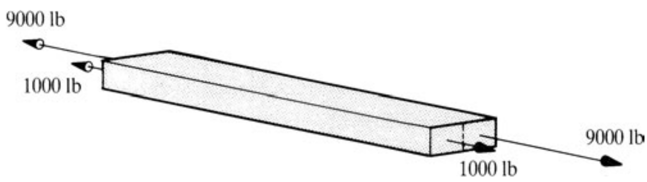


Figure 1.2. Illustration of the possibility of a nonuniform distribution of axial stresses across the cross-section of a bar.

that the stable loaded lengths of both bars are exactly the same. If the two bars are now fused together along their lengths to form one bar of cross-sectional area $2A$, while the respective loads are maintained on both halves of the now single bar, the result is a bar as shown in Fig. 1.2. In this case the bar is made of a single material with a stress acting over one half of the new cross-section that is nine times as great as that acting over the other half of the cross-section. When the stress does vary significantly, it is not useful to work with a value of the average stress over the entire cross-section. Clearly the more heavily loaded half of the fused bar is closer to rupture than the less loaded half. For that type of reason, engineers are usually much more interested in knowing the values of the peak stresses and knowing how extensive are the areas over which high stresses act. This latter information is much more useful when estimating the likelihood of local material failures or more general structural failures.

It is also important to note that simply stating that the two loads acting upon the combined bar's end cross-sectional area have a combined magnitude of 10 000 lbs would not be sufficiently informative with regard to the load distribution on the end surfaces of that bar. It will be necessary to be more precise when specifying the loads acting on the outer surfaces of the bodies under study.

The fact that stresses are not always constant over a given internal planar area requires careful consideration about how stress is to be defined. The definition must not compromise the basic concept of stress being the ratio of force over area. Since both force and area are measurable quantities, so then their ratio must also be a measurable quantity. Therefore it is necessary that the definition produce a unique measure; that is, that there be no ambiguity as to the magnitude of the stress. Consider Fig. 1.3(a). That sketch shows an edge view of a varying stress distribution acting over a beam cross-section that is positioned somewhere along the length of a loaded beam. (Beams have the same general geometry as bars, but the name "beam" indicates a more general type of loading.) Since an average stress value based on a total area is not a satisfactory measure, explore the possibility of a stress value that can be associated with smaller portions of the total area. A definition of stress based on sub-areas would permit having separate stress values where the stresses are high, or anywhere else. Therefore consider an arbitrarily located sub-area. Better yet, consider the arbitrarily located sequence of three sub-areas, from larger to smaller, as drawn in Fig. 1.3(b). Each sub-area in the sequence has been chosen as an included square, and each square has the point P as one vertex. Figure 1.3 (c) shows edge views of the stress distributions acting on that sequence of sub-areas. For the lack of a better idea, let it be said that the stress to be associated with each sub-area of the sequence is the average stress for that sub-area. Geometrically, the average stress is represented by the dashed line ordinate in Fig. 1.3(c). It is easily seen that in this case the value of the stress to be associated with each sub-area decreases as the area decreases. Similarly, if the sequence of squares approached point Q, instead of point P, the stress values would increase as the total force and the magnitude of the sub-area decreased. It also should be clear from the geometry that if the sequence of sub-area was greatly extended in an orderly fashion beyond three in number, the value of the average

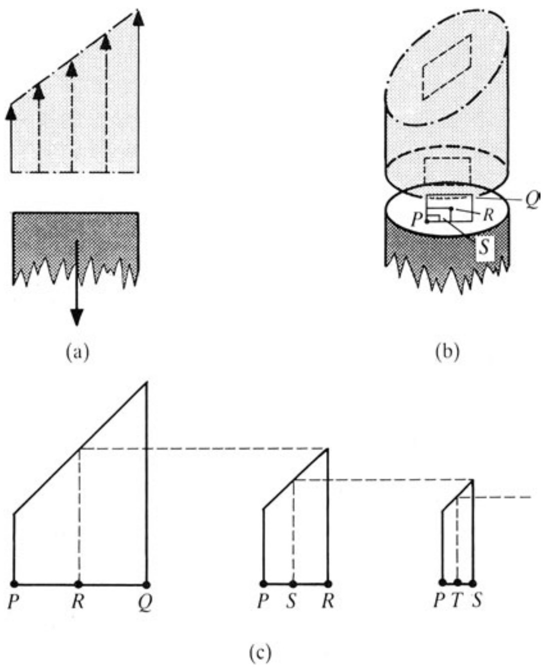


Figure 1.3. (a) Side view of a tensile force whose effect is distributed linearly over the bar cross-section. (b) The process of considering smaller and smaller portions of the bar cross-section at the fixed-point *P*. (c) A geometric illustration of how the average intensity of the distributed force near point *P* approaches a unique value as the small portion of the total cross-sectional area anchored at point *P* is systematically reduced by a factor of 4.

stress would stabilize (i.e., converge) as the sequence of dashed lines representing the stress magnitude approached either point P or Q. For example, for an approach to point P, the dashed line that depicts the average value of the stress over the sub-area in the sequence of sub-areas would irresistibly approach, and be confined by, the stress magnitude line at point P. This fixed stress magnitude at point P is precisely the unique force over area measure that is sought. This measure needs only to be expressed mathematically as the following limit,[1] where N, the total force acting over the sub-area A, is a function of A:

$$\text{Stress} \equiv \lim_{A \to 0} \frac{N}{A} \tag{1.1}$$

In this limit both the numerator and denominator decrease jointly to very small, even infinitesimal quantities.[2] Recall that the definition of a derivative is exactly the same type of limit. For example, the derivative of the function $f(x)$ at the point x_p is the limit as

[1] The three-bar symbol signifies that the relationship between the left-hand side and the right-hand side is that of an identity. An identity is an equality that is true in *all* circumstances. A simple way of appreciating the difference between an identity and a mere equality is to recall that for $0 \le \theta \le 2\pi$, the formula $\cos^2\theta + \sin^2\theta \equiv 1.0$ is true for all θ, while $\cos\theta + \sin\theta \equiv 1.0$ is only true for two values of θ. All definitions are identities.

[2] The atomic nature of materials is ignored in preference to the convenient fiction that all pure materials exhibit the same physical properties for small samples, no matter how small, as are exhibited on average for large samples of the material. This convenient approximation leads to the material being called a *continuum* and thus the material of Chapters 1–4 is called *solid mechanics*, a branch of continuum mechanics.

x approaches x_p of the ratio of $[f(x_p) - f(x)]/[x_p - x]$. Note how closely the above definition of the derivative fits the illustration in Fig. 1.3(c) where x_p is analogous to the point P, and x is analogous to the right-hand point in the sequence Q, R, S, ... This argument allows rewriting of the above definition, Eq. (1.1), as the ratio of two differentials; that is, as

$$\text{Stress} = \frac{dN}{dA} \tag{1.2}$$

This definition of stress is well and good as far as it goes, but it does not take into account the one further fact that the stress acting upon the cross-sectional area need not, as always assumed in the above discussion, act perpendicularly to the surface of the area under discussion. A simple demonstration that stress can act in the plane of the area as well as perpendicularly to the area occurs when one places one's hand firmly on a flat surface, and then rubs the surface with that hand, creating, by means of the friction between the hand and the surface, an in-plane stress on the flat surface. Note that the total force N acting upon the flat surface beneath the hand is the vector sum of the normal component (from pressing down with the hand) and the in-plane component. Since neither component is zero in this case, the total force vector N is neither normal to nor within the plane of the surface. Another confirmation of the possibility that the stress does not always act in a direction that is normal to the area under consideration can be obtained by merely passing an oblique plane through the first or second bar in Fig. 1.1(c). Since the total force, and hence the total stress, in the bar parallels the bar axis, and the normal to the oblique plane is not parallel to the bar axis, the stress is not normal to the oblique plane. Therefore, it is now necessary to adjust the above definition of stress to account for the directional properties of forces and stresses. When considering an area with a fixed orientation in space, stress is a vector quantity because it is a force vector (dN) divided by a scalar (dA). (The qualification "when considering an area with a fixed orientation" is important, and is developed later.)

On the basis of the above discussion, it is now a simple matter to define a *normal* stress as the limit of the ratio of the normal force acting upon an area of fixed orientation, as the magnitude of that area approaches zero. The same can be done for the in-plane stress, called the total *shearing stress*. This decomposition of the total stress into a normal stress and a total shear stress is significant because the effects of these two different types of stresses on materials can be quite different. Two more steps are necessary to make the above definitions still more useful. The first step is to introduce a coordinate system. To begin simply, consider a right-handed Cartesian[3] coordinate system where the x axis is normal to the area being studied, while the y and z axes lie in the tangent plane of the area under study. In this case the area is called an "x area," or "x surface," or "x plane," since the orientation of the plane is precisely located by its normal, the x axis. In other words, when the x axis is fixed in space, then any plane perpendicular to that axis is an x plane.

Lower-case sigma (σ) is chosen to symbolize stress. A double-subscript notation is used to identify which of the possible stress components is meant; see Fig. 1.4. The first subscript designates the plane of the area upon which the stress acts, while the second subscript designates the direction in which the stress acts. Looking at Fig. 1.4, for example,

$$\sigma_{xy} = \frac{dN_y}{dA_x} \tag{1.3}$$

[3] From Ref. [2], the adjective "Cartesian" is derived from the family name of René Descartes (1596–1650), who first introduced the coordinate method and established analytical geometry.