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Heights and measures on analytic spaces. A survey of recent results, and some remarks

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The first goal of this paper was to survey my definition in [19] of measures on non-archimedean analytic spaces in the sense of Berkovich and to explain its applications in Arakelov geometry. These measures are analogous the measures on complex analytic spaces given by products of first Chern forms of hermitian line bundles.[1] In both contexts, archimedean and non-archimedean, they are related with Arakelov geometry and the local height pairings of cycles. However, while the archimedean measures lie at the ground of the definition of the archimedean local heights in Arakelov geometry, the situation is reversed in the ultrametric case: we begin with the definition of local heights given by arithmetic intersection theory and define measures in such a way that the archimedean formulae make sense and are valid. The construction is outlined in Section 1, with references concerning metrized line bundles and the archimedean setting. More applications to Arakelov geometry and equidistribution theorems are discussed in Section 3.

The relevance of Berkovich spaces in Diophantine geometry has now made been clear by many papers; besides [19] and [20] and the general equidistribution theorem of Yuan [59], I would like to mention the works [38, 39, 40, 30] who discuss the function field case of the equidistribution theorem, as well as the potential theory on non-archimedean curves developed simultaneously by Favre, Jonsson & Rivera-Letelier

[1] M. Kontsevich and Yu. Tschinkel gave me copies of unpublished notes from the years 2000–2002 where they develop similar ideas to construct canonical non-archimedean metrics on Calabi–Yau varieties; see also [45, 46].

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[32, 33] and Baker & Rumely for the projective line [8], and in general by A. Thuillier's PhD thesis [55]. The reader will find many important results in the latter work, which unfortunately is still unpublished at the time of this writing.

Anyway, I found useful to add examples and complements to the existing (and non-) literature. This is done in Section 2. Especially, I discuss in Section 2.2 the relation between the reduction graph and the skeleton of a Berkovich curve, showing that the two constructions of measures coincide. Section 2.3 shows that the measures defined are of a local nature; more generally, we show that the measures vanish on any open subset where one of the metrized line bundles involved is trivial. This suggests a general definition of *strongly pluriharmonic* functions on Berkovich spaces, as uniform limits of logarithms of absolute values of invertible holomorphic functions. (Strongly pluriharmonic fonctions should only exhaust pluriharmonic functions when the residue field is algebraic over a finite field, but not in general.) In Section 2.4, we discuss polarized dynamical systems and explain the construction of canonical metrics and measures in that case. We also show that the canonical measure vanishes on the Berkovich equicontinuity locus. In fact, what we show is that the canonical metric is "strongly pluriharmonic" on that locus. This is the direct generalization of a theorem of [52] for the projective line (see also [8] for an exposition); this generalizes also a theorem of [44] that Green functions are locally constant on the classical equicontinuity locus. As already were their proofs, mine is a direct adaptation of the proof of the complex case [43]. In Section 2.5, following Gubler [41], we finally describe the canonical measures in the case of abelian varieties.

In Section 3, we discuss applications of the measures in Diophantine geometry over global fields. Once definitions are recalled out in Section 3.1, we briefly discuss in Section 3.2 the relation between Mahler measures (*i.e.*, integration of Green functions against measures) and heights. In Section 3.3, we survey the equidistribution theorems for Galois orbits of points of "small height", following the variational method of Szpiro–Ullmo–Zhang [54] and [59]. In fact, we describe the more general statement from [20]. Finally, Section 3.4 discusses positive lower bounds for heights on curves. This is inspired by recent papers [5, 49] but the method goes back to Mimar's unpublished thesis [48]. A recent preprint [58] of Yuan and Zhang establishes a similar result in any dimension.

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1 Metrized line bundles and measures

1.1 Continuous metrics

1.1.1 Definition. — Let X be a topological space together with a sheaf of local rings $\mathscr{O}_X$ ("analytic functions"); let also $\mathscr{C}_X$ be the sheaf of continuous functions on X. In analytic geometry, local functions have an absolute value which is a real valued continuous function, satisfying the triangle inequality. Let us thus assume that we have a morphism of sheaves $\mathscr{O}_X \to \mathscr{C}_X$, written $f \mapsto |f|$, such that $|fg| = |f|\,|g|$, $|1| = 1$, and $|f+g| \leq |f| + |g|$.

A line bundle on $(X, \mathscr{O}_X)$ is a sheaf L of $\mathscr{O}_X$-modules which is locally isomorphic to $\mathscr{O}_X$. In other words, X is covered by open sets U such that $\mathscr{O}_U \simeq L|U$; such an isomorphism is equivalent to a non-vanishing section $\varepsilon_U \in \Gamma(U, L)$, also called a local frame of L.

If s is a section of a line bundle L on an open set U, the value of s at a point $x \in U$ is only well-defined as an element of the stalk $L(x)$, which is a $\kappa(x)$-vector space of dimension 1. (Here, $\kappa(x)$ is the residue field of $\mathscr{O}_X$ at x.) Prescribing a metric on L amounts to assigning, in a coherent way, the norms of these values. Formally, a *metric* on L is the datum, for any open set $U \subset X$ and any section $s \in \Gamma(U, L)$, of a continuous function $\|s\|_U : U \to \mathbf{R}_+$, satisfying the following properties:

1. for any open set $V \subset U$, $\|s\|_V$ is the restriction to V of the function $\|s\|_U$;
2. for any function $f \in \mathscr{O}_X(U)$, $\|fs\| = |f| \, \|s\|$;
3. if s is a local frame on U, then $\|s\|$ doesn't vanish at any point of U.

One usually writes $\overline{L}$ for the pair $(L, \|\cdot\|)$ of a line bundle L and a metric on it.

Observe that the trivial line bundle $\mathscr{O}_X$ has a natural "trivial" metric, for which $\|1\| = 1$. In fact, a metric on the trivial line bundle $\mathscr{O}_X$ is equivalent to the datum of a continuous function h on X, such that $\|1\| = e^{-h}$.

1.1.2 The Abelian group of metrized line bundles. — Isomorphism of metrized line bundles are isomorphisms of line bundles which respect the metrics; they are called *isometries*. Constructions from tensor algebra extend naturally to the framework of metrized line bundles, compatibly with isometries. The tensor product of two metrized line bundles $\overline{L}$ and $\overline{M}$ has a natural metrization such that $\|s \otimes t\| = \|s\| \, \|t\|$, if s and t are local sections of L and M respectively. Similarly, the dual of a metrized line bundle has a metrization, and the obvious isomorphism $L \otimes L^\vee \simeq \mathscr{O}_X$ is an isometry. Consequently, isomorphism classes of metrized line bundles on X form an Abelian group $\overline{\mathrm{Pic}}(X)$. This group fits in an exact sequence

$$0 \to \mathscr{C}(X) \to \overline{\mathrm{Pic}}(X) \to \mathrm{Pic}(X),$$

where the first map associates to a real continuous function h on X the trivial line bundle endowed with the metric such that $\|1\| = e^{-h}$, and the second associates to a metrized line bundle the underlying line bundle. It is surjective when any line bundle has a metric (this certainly holds if X has partitions of unity).

Similarly, we can consider pull-backs of metrized line bundle. Let $\varphi \colon Y \to X$ be a morphism of locally ringed spaces such that $|\varphi^* f| = |f| \circ \varphi$ for any $f \in \mathscr{O}_X$. Let $\overline{L}$ be a metrized line bundle on X. Then, there is a canonical metric on $\varphi^* L$ such that $\|\varphi^* s\| = \|s\| \circ \varphi$ for any section $s \in \Gamma(U, L)$. This induces a morphism of Abelian groups $\varphi^* \colon \overline{\mathrm{Pic}}(X) \to \overline{\mathrm{Pic}}(Y)$.

1.2 The case of complex analytic spaces

1.2.1 Smooth metrics. — In complex analytic geometry, metrics are a very well established tool. Let us first consider the case of the projective space $X = \mathbf{P}^n(\mathbf{C})$; a point $x \in X$ is a $(n+1)$-tuple of homogeneous coordinates $[x_0 : \cdots : x_n]$, not all zero, and up to a scalar. Let $\pi\colon \mathbf{C}_*^{n+1} \to X$ be the canonical projection map, where the index $*$ means that we remove the origin $(0, \dots, 0)$. The fibers of π have a natural action of $\mathbf{C}^*$ and the line bundle $\mathscr{O}(1)$ has for sections s over an open set $U \subset \mathbf{P}^n(\mathbf{C})$ the analytic functions F_s on the open set $\pi^{-1}(U) \subset \mathbf{C}_*^{n+1}$ which are homogeneous of degree 1. The *Fubini-Study metric* of $\mathscr{O}(1)$ assigns to the section s the norm $\|s\|_{\mathrm{FS}}$ defined by

$$\|s\|_{\mathrm{FS}}([x_0 : \cdots : x_n]) = \frac{|F_s(x_0, \dots, x_n)|}{\left(|x_0|^2 + \cdots + |x_n|^2\right)^{1/2}}.$$

It is more than continuous; indeed, if s is a local frame on an open set U, then $\|s\|$ is a $\mathscr{C}^\infty$-function on U; such metrics are called *smooth.*

1.2.2 Curvature. — Line bundles with smooth metrics on smooth complex analytic spaces allow to perform differential calculus. Namely, the *curvature* of a smooth metrized line bundle $\overline{L}$ is a differential form $c_1(\overline{L})$ of type $(1,1)$ on X. Its definition involves the differential operator

$$\mathrm{dd}^c = \frac{i}{\pi}\partial\overline{\partial}.$$

When an open set $U \subset X$ admits local coordinates $(z_1, \dots, z_n)$, and $s \in \Gamma(U, L)$ is a local frame, then

$$c_1(\overline{L})|_U = \mathrm{dd}^c \log \|s\|^{-1} = \frac{i}{\pi} \sum_{1 \le j,k \le n} \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log \|s\|^{-1} \, \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k.$$

The Cauchy-Riemann equations ($\partial f/\partial \overline{z} = 0$ for any holomorphic function f of the variable z) imply that this formula does not depend on the choice of a local frame s. Consequently, these differential forms defined locally glue to a well-defined global differential form on X.

Taking the curvature form of a metrized line bundle is a linear operation: $c_1(\overline{L} \otimes \overline{M}) = c_1(\overline{L}) + c_1(\overline{M})$. It also commutes with pull-back: if $f\colon Y \to X$ is a morphism, then $f^* c_1(\overline{L}) = c_1(f^*\overline{L})$.

In the case of the Fubini-Study metric over the projective space $\mathbf{P}^n(\mathbf{C})$, the curvature is computed as follows. The open subset U_0 where the homogeneous coordinate x_0 is non-zero has local coordinates $z_1 = x_1/x_0, \dots, z_n = x_n/x_0$; the homogeneous polynomial X_0 defines a non-vanishing section s_0 of $\mathscr{O}(1)$ on U_0 and

$$\log \|s_0\|_{\mathrm{FS}}^{-1} = \frac{1}{2}\log\left(1+\sum_{j=1}^n |z_j|^2\right).$$

Consequently, over U_0,

$$\begin{aligned} c_1(\overline{\mathscr{O}(1)}_{\mathrm{FS}}) &= \frac{i}{\pi}\partial\overline{\partial}\log\|s_0\|_{\mathrm{FS}}^{-1} \\ &= \frac{i}{2\pi}\partial\left(\sum_{k=1}^n \frac{z_k}{1+\|z\|^2}\mathrm{d}\overline{z}_k\right) \\ &= \frac{i}{2\pi}\sum_{j=1}^n \frac{1}{1+\|z\|^2}\mathrm{d}z_j\wedge\mathrm{d}\overline{z}_j - \frac{i}{2\pi}\sum_{j,k=1}^n \frac{z_k\overline{z}_j}{(1+\|z\|^2)^2}\mathrm{d}z_j\wedge\mathrm{d}\overline{z}_k. \end{aligned}$$

In this calculation, we have abbreviated $\|z\|^2 = \sum_{j=1}^n |z_j|^2$.

1.2.3 Products, measures. — Taking the product of n factors equal to this differential form, we get a differential form of type (n,n) on the n-dimensional complex space X. Such a form can be integrated on X and the Wirtinger formula asserts that

$$\int_{\mathrm{X}} c_1(\overline{L})^n = \deg(L)$$

is the *degree* of L as computed by intersection theory. As an example, if $\mathrm{X} = \mathbf{P}^1(\mathbf{C})$, we have seen that

$$c_1(\overline{\mathscr{O}(1)}_{\mathrm{FS}}) = \frac{i}{2\pi(1+|z|^2)^2}\mathrm{d}z\wedge d\overline{z},$$

where $z = x_1/x_0$ is the affine coordinate of $\mathrm{X}\setminus\{\infty\}$. Passing in polar coordinates $z = re^{i\theta}$, we get

$$c_1(\overline{\mathscr{O}(1)}_{\mathrm{FS}}) = \frac{1}{2\pi(1+r^2)^2}2r\mathrm{d}r\wedge\mathrm{d}\theta$$

whose integral over $\mathbf{C}$ equals

$$\int_{\mathbf{P}^1(\mathbf{C})} c_1(\overline{\mathscr{O}(1)}_{\mathrm{FS}}) = \int_0^\infty \frac{1}{2\pi(1+r^2)^2}2r\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta = \int_0^\infty \frac{1}{(1+u)^2}\mathrm{d}u = 1.$$

1.2.4 The Poincaré–Lelong equation. — An important formula is the Poincaré–Lelong equation. For any line bundle L with a smooth metric, and any section $s \in \Gamma(\mathrm{X}, L)$ which does not vanish identically on any connected component of X, it asserts the following equality of *currents*[2]:

$$\mathrm{dd}^c \log \|s\|^{-1} + \delta_{\mathrm{div}(s)} = c_1(\overline{L}),$$

where $\mathrm{dd}^c \log \|s\|^{-1}$ is the image of $\log \|s\|^{-1}$ under the differential operator dd^c, taken in the sense of distributions, and $\delta_{\mathrm{div}(s)}$ is the current of integration on the cycle $\mathrm{div}(s)$ of codimension 1.

1.2.5 Archimedean height pairing. — Metrized line bundles and their associated curvature forms are a basic tool in Arakelov geometry, invented by Arakelov in [2] and developped by Faltings [31], Deligne [25] for curves, and by Gillet-Soulé [34] in any dimension. For our concerns, they allow for a definition of height functions for algebraic cycles on algebraic varieties defined over number fields. As explained by Gubler [35, 36], they also permit to develop a theory of archimedean local heights.

For simplicity, let us assume that X is proper, smooth, and that all of its connected components have dimension n.

Let $\overline{L}_0, \dots, \overline{L}_n$ be metrized line bundles with smooth metrics. For $j \in \{0, \dots, n\}$, let s_j be a regular meromorphic section of L_j and let $\mathrm{div}(s_j)$ be its divisor. The given metric of L_j furnishes moreover a function $\log \|s_j\|^{-1}$ on X and a $(1,1)$-form $c_1(\overline{L}_j)$, related by the Poincaré–Lelong equation $\mathrm{dd}^c \log \|s_j\|^{-1} + \delta_{\mathrm{div}(s_j)} = c_1(\overline{L}_j)$. In the terminology of Arakelov geometry, $\log \|s_j\|^{-1}$ is a Green current (here, function) for the cycle $\mathrm{div}(s_j)$; we shall write $\widehat{\mathrm{div}}(s_j)$ for the pair $(\mathrm{div}(s_j), \log \|s_j\|^{-1})$.

Let $\mathrm{Z} \subset \mathrm{X}$ be a k-dimensional subvariety such that the divisors $\mathrm{div}(s_j)$, for $0 \leq j \leq k$, have no common point on Z. Then, one defines inductively the local height pairing by the formula:

$$(\widehat{\mathrm{div}}(s_0) \dots \widehat{\mathrm{div}}(s_k) | \mathrm{Z}) = (\widehat{\mathrm{div}}(s_0) \dots \widehat{\mathrm{div}}(s_{k-1}) | \mathrm{div}(s_k|_{\mathrm{Z}})) + \int_{\mathrm{X}} \log \|s_k\|^{-1} c_1(\overline{L}_0) \dots c_1(\overline{L}_{k-1}) \delta_{\mathrm{Z}}. \quad (9.1)$$

[2] The space of currents is the dual to the space of differential forms, with the associated grading; in the orientable case, currents can also be seen as differential forms with distribution coefficients.

The second hand of this formula requires two comments. 1) The divisor $\operatorname{div}(s_k|_{\mathrm{Z}})$ is a formal linear combination of $(k-1)$-dimensional subvarieties of X, and its local height pairing is computed by linearity from the local height pairings of its components. 2) The integral of the right hand side involves a function with singularities ($\log \|s_k\|^{-1}$) to be integrated against a distribution: in this case, this means restricting the differential form $c_1(\overline{L}_0)\dots c_1(\overline{L}_{k-1})$ to the smooth part of Z, multiplying by $\log \|s_k\|^{-1}$, and integrating the result. The basic theory of closed positive currents proves that the resulting integral converges absolutely; as in [34], one can also resort to Hironaka's resolution of singularities.

It is then a non-trivial result that the local height pairing is symmetric in the involved divisors; it is also multilinear. See [37] for more details, as well as [34] for the global case.

1.2.6 Positivity. — Consideration of the curvature allows to define positivity notions for metrized line bundles. Namely, one says that a smooth metrized line bundle $\overline{L}$ is *positive* (resp. *semi-positive*) if its curvature form is a positive (resp. a non-negative) $(1,1)$-form. This means that for any point $x \in \mathrm{X}$, the hermitian form $c_1(\overline{L})_x$ on the complex tangent space $\mathrm{T}_x\mathrm{X}$ is positive definite (resp. non-negative). As a crucial example, the line bundle $\mathscr{O}(1)$ with its Fubini-Study metric is positive. The pull-back of a positive metrized line bundle by an immersion is positive. In particular, ample line bundles can be endowed with a positive smooth metric; Kodaira's embedding theorem asserts the converse: if a line bundle possesses a positive smooth metric, then it is ample.

The pull-back of a semi-positive metrized line bundle by any morphism is still semi-positive. If $\overline{L}$ is semi-positive, then the measure $c_1(\overline{L})^n$ is a positive measure.

1.2.7 Semi-positive continuous metrics. — More generally, both the curvature and the Poincaré–Lelong equation make sense for metrized line bundles with arbitrary (continuous) metrics, except that $c_1(\overline{L})$ has to be considered as a current. The notion of semi-positivity can even be extended to this more general case, because it can be tested by duality: a current is positive if its evaluation on any nonnegative differential form is nonnegative. Alternatively, semi-positive (continuous) metrized line bundles are characterized by the fact that for any local frame s of $\overline{L}$ over an open set U, the continuous function $\log \|s\|^{-1}$ is *plurisubharmonic* on U. In turn, this means that for any morphism $\varphi\colon \overline{D} \to \mathrm{U}$, where

$\overline{D} = \overline{D}(0,1)$ is the closed unit disk in $\mathbf{C}$,

$$\log \|s\|^{-1}(\varphi(0)) \leq \frac{1}{2\pi}\int_0^{2\pi} \log \|s\|^{-1}(\varphi(e^{i\theta}))\mathrm{d}\theta.$$

Assume that $\overline{L}$ is semi-positive. Although products of currents are not defined in general (not more than products of distributions), the theory of Bedford–Taylor [10, 9] and Demailly [26, 27] defines a current $c_1(\overline{L})^n$ which then is a positive measure on X. There are two ways to define this current. The first one works locally and proceeds by induction: if $u = \log \|s\|^{-1}$, for a local non-vanishing section s of L, one defines a sequence (T_k) of closed positive currents by the formulae $T_0 = 1$, $T_1 = \mathrm{dd}^c u, \ldots, T_{k+1} = \mathrm{dd}^c(uT_k)$ and $c_1(\overline{L})^n = \mathrm{dd}^c(u)^n$ is defined to be T_n. What makes this construction work is the fact that at each step, uT_k is a well-defined current (product of a continuous function and of a positive current), and one has to prove that T_{k+1} is again a closed positive current. The other way, which shall be the one akin to a generalization in the ultrametric framework, consists in observing that if L is a line bundle with a continuous semi-positive metric $\|\cdot\|$, then there exists a sequence of smooth semi-positive metrics $\|\cdot\|_k$ on the line bundle L which converges uniformly to the initial metric: for any local section s, $\|s\|_k$ converges uniformly to $\|s\|$ on compact sets. The curvature current $c_1(\overline{L})$ is then the limit of the positive currents $c_1(\overline{L}_k)$, and the measure $c_1(\overline{L})^n$ is the limit of the measures $c_1(\overline{L}_k)^n$. (We refer to [47] for the global statement; to construct the currents, one can in fact work locally in which case a simple convolution argument establishes the claim.)

An important example of semi-positive metric which is continuous, but not smoth, is furnished by the Weil metric on the line bundle $\mathscr{O}(1)$ on $\mathbf{P}^n(\mathbf{C})$. This metric is defined as follows: if $\mathrm{U} \subset \mathbf{P}^n(\mathbf{C})$ is an open set, and s is a section of $\mathscr{O}(1)$ on U corresponding to an analytic function F_s on $\pi^{-1}(\mathrm{U}) \subset \mathbf{C}_*^{n+1}$ which is homogeneous of degree 1, then for any $(x_0, \ldots, x_n) \in \pi^{-1}(\mathrm{U})$, one has

$$\|s\|_{\mathrm{W}} = \frac{|F_s(x_0, \ldots, x_n)|}{\max(|x_0|, \ldots, |x_n|)}.$$

The associated measure $c_1(\overline{\mathscr{O}(1)}_{\mathrm{W}})^n$ on $\mathbf{P}^n(\mathbf{C})$ is as follows, cf. [62, 47]: the subset of all points $[x_0 : \cdots : x_n] \in \mathbf{P}^n(\mathbf{C})$ such that $|x_j| = |x_k|$ for all j, k is naturally identified with the polycircle $\mathbf{S}_1^n$ (map $[x_0 : \cdots : x_n]$ to $(x_1/x_0, \ldots, x_n/x_0)$); take the normalized Haar measure of this compact group and push it onto $\mathbf{P}^n(\mathbf{C})$.

1.2.8 Admissible metrics. — Let us say that a continuous metrized line bundle is *admissible* if it can be written as $\overline{L} \otimes \overline{M}^{\vee}$, where $\overline{L}$ and $\overline{M}$ are metrized line bundles whose metrics are continuous and semi-positive. Admissible metrized line bundles form a subgroup $\overline{\mathrm{Pic}}_{\mathrm{ad}}(\mathrm{X})$ of $\overline{\mathrm{Pic}}(\mathrm{X})$ which maps surjectively onto $\mathrm{Pic}(\mathrm{X})$ if X is projective.

The curvature current $c_1(\overline{L})$ of an admissible metrized line bundle $\overline{L}$ is a differential form of type $(1,1)$ whose coefficients are signed measures. Its nth product $c_1(\overline{L})^n$ is well-defined as a signed measure on X.

1.2.9 Local height pairing (admissible case). — The good analytic properties of semi-positive metrics allow to extend the definition of the local height pairing to the case of admissible line bundles. Indeed, when one approximates uniformly a semi-positive line bundle by a sequence of smooth semi-positive line bundles, one can prove that the corresponding sequence of local height pairings converges, the limit being independent on the chosen approximation.

The proof is inspired by Zhang's proof of the global case in [63] and goes by induction. Let us consider, for each j, two smooth semi-positive metrics on the line bundle L_j and assume that they differ by a factor e^{-h_j}. Then, the corresponding local height pairings differ from an expression of the form

$$\sum_{j=0}^{k} \int_{\mathrm{Z}} h_j c_1(\overline{L}_0) \dots \widehat{c_1(\overline{L}_j)} \dots c_1(\overline{L}_k),$$

where the written curvature forms are associated to the first metric for indices $< j$, and to the second for indices $> j$. This differential forms are positive by assumption, so that the integral is bounded in absolute value by

$$\sum_{j=0}^{k} \|h_j\|_{\infty} \int_{\mathrm{Z}} c_1(\overline{L}_0) \dots \widehat{c_1(\overline{L}_j)} \dots c_1(\overline{L}_k)$$
$$= \sum_{j=0}^{K} \|h_j\|_{\infty} \, (c_1(L_0) \dots \widehat{c_1(L_j)} \dots c_1(L_k)|\mathrm{Z}),$$

where the last expression is essentially a degree. (In these formulae, the factor with a hat is removed.) This inequality means that on the restriction to the space of smooth semi-positive metrics, with the topology of uniform convergence, the local height pairing is uniformly continuous.