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Automorphisms and Equivalence Relations in Topological Dynamics

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To Valerie, Carrie, and Kathleen; in memory of Betty.

Contents

Introduction

To a large extent this book is an updated version of *Lectures on Topological Dynamics* by Robert Ellis [Ellis, R., (1969)]. That book gave an exposition of what might be called an algebraic theory of minimal sets. Our goal here is to give a clear, self contained exposition of a new approach to the theory which allows for more straightforward proofs and develops a clearer language for expressing many of the fundamental ideas. We have included a treatment of many of the results in the aforementioned exposition, in addition to more recent developments in the theory; we have not attempted, however, to give a complete or exhaustive treatment of all the known results in the algebraic theory of minimal sets. Our hope is that the reader will be motivated to use the language and techniques to study related topics not touched on here. Some of these are mentioned either in the exercises or notes given at the end of various sections. This book should be suitable for a graduate course in topological dynamics whose prerequisites need only include some background in topology. We assume the reader is familiar with compact Hausdorff spaces, convergence of nets, etc., and perhaps has had some exposure to uniform structures and pseudo metrics which play a limited role in our exposition.

A *flow* is a triple (X, T, π) where X is a compact Hausdorff space, T is a topological group, and $\pi : X \times T \to X$ is a continuous action of T on X, so that $xe = x$ and $(xt)s = x(ts)$ for all $x \in X$, $s, t \in T$. Here we write $xt = \pi(x, t)$ for all $x \in X$ and $t \in T$, and e is the identity of the group T. Usually the symbol π will be omitted and the flow (X, T, π) denoted by (X, T) or simply by X. In the situations considered here there is no loss of generality if T is given the discrete topology. The assumptions made thus far do not suffice to produce an interesting theory. The group T may be too "small" in its action on X. Thus for example, the trivial case where $xt = x_0$, a fixed element of X, for all $x \in X$ and $t \in T$, is not ruled out. To eliminate such degenerate behavior it is convenient to assume that the flow (X, T) is *point* *transitive*, i.e. that there exists $x_0 \in X$ such that its orbit $x_0T \equiv \{x_0t \mid t \in T\}$ is dense in X.

The category $\mathcal{P}$ of point transitive flows has the remarkable property that it possesses a universal object; i,e, there exists a point transitive flow $(\beta T, T)$ such that any flow in $\mathcal{P}$ is a homomorphic image of βT. (See section 1 for a description of $(\beta T, T)$ and the proof of its universality.) Moreover, one may associate in a canonical fashion with any flow (X, T) a point transitive flow $E(X, T)$. The latter, called the *enveloping semigroup* has proved extremely useful in the study of the dynamical properties of the original flow (X, T). The enveloping semigroup is defined and studied in section 2, and examples of its use are scattered throughout the subsequent sections.

This exposition focuses, however, on the category, $\mathcal{M}$, of *minimal* flows. These are flows for which the orbit of every point $x \in X$ is dense; that is $\overline{xT} = \overline{\{xt \mid t \in T\}} = X$ for all $x \in X$. Again there exists a (unique up to isomorphism) universal object M in $\mathcal{M}$. This fact was exploited in several papers to develop an "algebraic theory" of minimal flows. In particular a group was associated with each such flow and various relations among minimal flows studied by means of these groups. One purpose of this volume is to collect in one place the techniques which have proved useful in this study; another goal is to provide an exposition of a new approach to this material.

The account of this algebraic theory of minimal flows given in *Lectures on Topological Dynamics* depends heavily on an algebraic point of view derived by studying the collection $\mathcal{C}(X)$ of continuous functions on X rather than X itself. In this volume we instead exploit the fact that X, as a homomorphic image of M, is of the form M/R for some *icer* (invariant closed equivalence relation) on M. We study the flow (X, T) via the icer R rather than the algebra $\mathcal{C}(X)$.

Another change is that the role of the group of automorphisms of a flow is emphasized. In particular the group G of automorphisms of M plays a crucial role. It is used both to codify the algebraic structure of M, and to define the groups associated to the minimal flows in $\mathcal{M}$. In the earlier approach G was viewed as a subset Mu of M, where $u \in M$ was a fixed idempotent. The new approach eliminates the asymmetrical treatment of the idempotents. Instead we view $M = \biguplus G(u)$ as a disjoint union (taken over all the idempotents in $u \in M$) of the images of the idempotents under the group G. Thus we explicitly take advantage of the fact that every $p \in M$ can be written uniquely in the form $\alpha(u)$ with $\alpha \in G$ and u an idempotent in M. This approach also makes reliance on the concept of a pointed flow unnecessary. Previously the concept of a pointed flow was used to define, up to conjugacy, the group of a minimal flow; a different choice of base point corresponding to a conjugate subgroup of Mu. From the point of view of icers on M, the group of the flow M/R is the subgroup:

$$G(R) = \{\alpha \in G \mid gr(\alpha) \subset R\},$$

of G. Here $gr(\alpha) = \{(p, \alpha(p)) \mid p \in M\}$ is the graph of the automorphism α of M. Again if S is an icer with $M/R \cong M/S$, then $G(S)$ is conjugate to $G(R)$.

One of the important tools for the study of minimal flows is the so-called τ-topology on G. In section 10 we show how one can define a topology on the automorphism group $Aut(X)$ of any regular flow (X, T). Since $G = Aut(M)$ and (M, T) is regular, this allows one to define a topology on G. This topology on G coincides with the original definition of the τ-topology. (The idea for this viewpoint stems from J. Auslander's approach to the τ-topology–private communication.)

We would now like to make a few comments on some of the results which have been included herein. In part I we lay the foundation for what follows by treating the universal constructions upon which much of the later material is based. This includes an introduction to βT, the enveloping semigroup, and the universal minimal flow. The flow $(2^X, T)$ whose minimal subflows are the so-called quasi-factors of the minimal flow (X, T) is discussed in section 5. Here 2^X is the collection of non-empty closed subsets of X. The space 2^X is given the Vietoris topology detailed in the appendix to section 5. The extension of the action of T on 2^X to an action of βT on 2^X via the circle operator is also discussed in section 5, and used later in sections 12 and 17.

Part II develops many of the techniques and language critical to our approach. As mentioned above, this approach hinges on identifying minimal flows as quotients of M by icers. We need not only to associate to any minimal flow an icer on M, but to any icer on M a minimal flow. The basic topological result needed is that the quotient of any compact Hausdorff space by a closed equivalence relation is again a compact Hausdorff space. Section 6 includes a proof of this result and a discussion of the relative product of two relations, a useful tool for constructing equivalence relations.

The fundamental result concerning icers on M is proven in section 7. We show that any icer R on M can be written as a relative product

$$R = (R \cap P_0) \circ gr(G(R))$$

where $P_0 = \{(\alpha(u), \alpha(v)) \mid \alpha \in G \text{ and } u, v \text{ are idempotents in } M\}$ (see **7.21**). Regular flows, whose original definition is motivated in terms of automorphisms, are those flows whose representation as a quotient M/R is unique. The flow (M, T) is of course regular, and its structure serves as a prototype for the algebraic structure of regular flows outlined in section 8. In particular, if (X, T) is a regular flow, then the pair $\{X, Aut(X)\}$ has properties analogous to those of the pair $\{M, G\}$, some of which were alluded to above.

In part III we give a detailed exposition of the approach to the τ-topology mentioned earlier. When applied to the group $Aut(X)$, for any regular minimal flow (X, T), we obtain a topology which is compact and T_1 but not Hausdorff. The construction of a *derived group* F' for any closed subgroup $F \subset Aut(X)$ is given in section 11. F' is a normal subgroup of F which measures the extent to which F fails to be Hausdorff; in fact for any closed subgroup $H \subset F$, the quotient space F/H is Hausdorff if and only if $F' \subset H$ (see **11.10**). In section 12 we give a proof of the fact that there exists a minimal flow X whose group $G(X) = A$ if and only if A is a τ-closed subgroup of G. One example of such a flow is M/R where

$$R = \overline{gr(A)} = \overline{\bigcup\{gr(\alpha) \mid \alpha \in A\}}.$$

The basic idea of the proof, which uses the material on quasi-factors, is the same as in *Lectures on Topological Dynamics* but the language of the current approach allows a more efficient treatment.

Part IV is motivated by the questions: How are the various subgroups of G related to one another, and what do they tell us about the dynamics of minimal flows? It has long been known that the subcategories $\mathcal{D}$ and $\mathcal{E}$ of minimal distal and minimal equicontinuous flows respectively also possess universal objects $X_{\mathcal{D}}$ and $X_{\mathcal{E}}$. Heretofore the groups D and E have been defined as the groups associated to these flows, i.e. $D = G(X_{\mathcal{D}})$ and $E = G(X_{\mathcal{E}})$. In sections 14 and 15 we obtain intrinsic characterizations of D (see **14.6**), and E (see **15.23**) respectively. This gives content to the statements: if X is distal, then $D \subset G(X)$, and if X is equicontinuous, then $E \subset G(X)$. In fact, emphasizing the language of icers, M/R is distal (respectively equicontinuous) if and only if

$$R = P_0 \circ gr(A)$$

with $D \subset A$ (respectively $E \subset A$). For proofs see **14.10** and **15.14** respectively. In particular distal and equicontinuous flows are completely determined by their groups. We show in **15.21** that $G'D = E$, from which it follows immediately that (X, T) is equicontinuous if and only if (X, T) is distal and $G' \subset G(X)$. In section 13 we discuss the proximal relation $P(X)$ on a minimal flow X. In analogy with the distal and equicontinuous cases, we give a description of a subgroup $P \subset G$ and show that $P(X)$ is an equivalence relation if and only if $P \subset G(X)$. Another subgroup $G^J \subset G$ is introduced and we show that $P(X)$ is an equivalence relation with closed cells if and only if $PG^J \subset G(X)$. In fact $PG^J \underset{\neq}{\subset} D$ which is consistent with the well-known result that $P(X)$ is a closed invariant equivalence relation on X if and only if $D \subset G(X)$. (see **14.8**) In section 15 the regionally proximal relation, $Q(X)$ of a minimal flow (X, T) is introduced to facilitate the study of equicontinuous flows. (Recall that (X, T) is equicontinuous if and only if $Q(X) = \Delta_X$ the diagonal in $X \times X$.) The case $Q \equiv Q(M)$ is also used to define the group E. Equicontinuous minimal flows are discussed from the point of view of icers on M in the body of section 15, while the approach to the same material via the enveloping semigroup is treated in the appendix. $Q(X)$ is discussed in further detail in section 15 where we give a new proof of the fact that if $E \subset G'G(X)$, then $Q(X)$ is an equivalence relation.

To a large extent part V is concerned with generalizing the results of part IV to homomorphisms (extensions) of minimal flows. For instance for icers $R \subset S$ on M, the canonical projection $M/R \to M/S$ is a distal homomorphism if and only if

$$S = (R \cap P_0) \circ gr(G(S)),$$

moreover the extension is equicontinuous if and only if $G(S)' \subset G(R)$. We close with a section devoted to four theorems all of which are equivalent to the Furstenberg structure theorem for distal extensions; this section uses the language of icers and the techniques developed in the earlier sections to give proofs that all four theorems are equivalent. This fact does not seem to have been emphasized in the literature, and provides a good opportunity to illustrate the language and techniques developed in the book. This analysis also illustrates an interesting twist to the icer approach. Here not only does the structure of the icers R and S come into play in understanding the extension $M/R \to M/S$, the *dynamics* of the *icer* on M/R whose quotient gives M/S also plays an important role. The construction of the so-called Furstenberg tower provides another nice illustration of the language of icers; the stages in the tower are explicitly constructed using icers which are themselves constructed from the groups involved.

Section 20 itself does not contain the proof of the Furstenberg theorem. Instead we give a chart describing where proofs of various special cases appear in the text. On the other hand a complete proof for compact Hausdorff spaces, of the fact that any icer on a minimal flow which is both topologically transitive and pointwise almost periodic must be trivial (one of the equivalents of the Furstenberg structure theorem) appears in **9.13**. This is because our proof relies on the concept of the quasi-relative product developed in section 9. Indeed the quasi-relative product arose during our attempt to give a proof of the Furstenberg theorem in its full generality. The metric case of the theorem follows immediately from the fact that for metric flows the notions of point-transitivity and topological transitivity coincide. Our proof in the general case proceeds by reducing it to the metric case; the key tool in the construction which enables this is the quasi-relative product. While the quasi-relative product is only necessary for the most general version of the Furstenberg theorem, it turns out to be closely connected to quasi-factors (hence the name) and RIC extensions. We detail these connections in sections 9 and 17 respectively.

A word about format

We have written this book using a theorem-proof format. All the proofs are given using a sequence of numbered steps for which reasons are given at each stage. There are two main reasons for this approach. The first is to make sure that the arguments are as clear and accessible as possible. We found that insisting on numbering our steps and giving reasons forced a rigor, clarity, and attention to detail we hope the reader will appreciate. We have attempted to avoid situations where as the material becomes more complex the reader is expected to fill in more gaps in the arguments.

In addition to a better understanding of the details of the individual arguments, we hope that the format adds to the clarity of the overall exposition. The assumptions and conclusions of each of the lemmas, propositions, and theorems are stated carefully and precisely in a consistent format. These items are all numbered so that they can be referred to in a precise and unambiguous way as the exposition proceeds. We have tried to keep the proofs reasonably short and have divided the material into short sections, typically ten to fifteen pages long. In addition, we begin each section with an introduction designed to give an informal outline and motivation for the material in that section. The reader who wishes to go lightly on the intricate details, may wish to follow the train of thought by focusing on the introductions to each section and skipping the proofs. In this case, if a specific result attracts the reader's interest, then the numbering system should facilitate a more careful reading of the details. This format is designed especially for the student who is not yet an expert; it assures that careful attention is paid to the details and that the train of thought is readily accessible.