Cambridge University Press
978-1-107-62266-1 - Proceedings of the International Congress of Mathematicians: 14–21 August 1958
Edited by J. A. Todd
Excerpt
More information

ONE HOUR ADDRESSES

MODERN DEVELOPMENT OF SURFACE THEORY

By A. D. ALEXANDROV

I will try to give in my lecture an outline of a general theory of surfaces as it has been developed during the past decade by a group of Russian geometers, U. F. Borisov, V. A. Zalgaller, A. V. Pogorelov, U. G. Reshetnak, I. J. Backelman, V. V. Streltsov and myself. This theory arose as a natural generalization of the theory of convex surfaces, the systematic presentation of which was given in my book *The Intrinsic Geometry of Convex Surfaces*, published just 10 years ago. Now this general theory has grown into an extensive branch of geometry with its own concepts, problems, methods and numerous results.

It would be hopeless to try to give here more than a general idea of the theory, so that all details and many results even of a fundamental character must be omitted. In constructing the foundations of the theory my aim was to define and to study the most general surfaces which allow of concepts and results analogous to those of classic Gaussian theory of surfaces. There are, first of all, two basic concepts of Gaussian theory; that of the intrinsic metric of a surface and that of its curvature. We accept an integral point of view according to which the metric is determined not by means of a line-element but by means of the distances between points measured *in* the surface, and the curvature is considered as a set-function, so that we mean integral curvature of a domain on the surface instead of the curvature at a point.

Let a surface S possess the property that any two of its points x, y can be joined by a curve $\widehat{xy}$ which lies in S and has a finite length $s(\widehat{xy})$. We define the intrinsic distance as

$$\rho_s(x,y) = \inf_{\widehat{xy} \subset S} s(\widehat{xy}). \tag{1}$$

It is evident that it satisfies the usual conditions imposed upon the general concept of a metric:

$$(1)\ \ \rho(x,y) = 0 \text{ if and only if } x = y;$$

$$(2)\ \ \rho(x,y)+\rho(y,z) \geqslant \rho(z,x).$$

Thus the surface becomes a metric space with the metric ρ_s.

There are two points of view; one may consider a surface as a figure in the space, being interested in its special shape; or the surface may be considered as a metric space with the intrinsic metric ρ. In this case we speak of the intrinsic geometry of the surface, while in the first case we speak of its external geometry.

Corresponding to these two points of view there are two concepts of the curvature of a surface. The external curvature is measured by means of the area of the spherical image and the intrinsic curvature is measured by means of the excesses of geodesic triangles, the excess of a triangle T with the angles α, β, γ being, by definition,

$$\delta(T) = \alpha+\beta+\gamma-\pi.$$

As we are going to consider surfaces with a definite, i.e. finite, curvature we speak of surfaces with bounded curvature. Thus the objects of the purely intrinsic theory are two-dimensional metric manifolds with bounded intrinsic curvature (M.B.C.) and the objects of the external theory are surfaces with bounded curvature (S.B.C.).

The intrinsic and the external theories are not independent; there exists a close connection between them and, first of all, the well-known theorem by Gauss which asserts the equality of the intrinsic and the external curvatures for, at least, sufficiently regular surfaces. Thus we have sketched a certain programme; to give the strict definitions of manifolds of bounded curvature and of surfaces of bounded external curvature, to study their properties and to establish the connection between the intrinsic and the external properties of these surfaces.

A somewhat different, and in some respects even more general, approach to the theory of surfaces may be based upon the concept of parallel translation, which is closely connected with the concept of curvature because of the well-known Gauss–Bonnet theorem. The parallel translation of a vector along a curve on a surface can be defined both in intrinsic and external terms by means of the Levi-Civita construction. If we follow this trend of ideas the objects of the theory are the metric manifolds and the surfaces where parallel translation of vectors is defined for a sufficiently ample set of curves. Such surfaces were studied recently by Borisov, and I will give an account of his results in the last part of my lecture.

1. The definition of M.B.C.

1.1. The intrinsic metric. The length of a curve is defined in any metric space in the usual manner; it is the supremum of the sums of the distances between successive points of the curve. If any two points of a set S in a metric space can be joined by a curve which lies in S and has finite length, we call the set metrically connected. Now we say that the metric of a space is intrinsic in itself, or, simply, intrinsic, provided that the space is metrically connected and the distance between any two points is equal to the infimum of the lengths of curves joining these points.

The introduction of this concept is justified by the following theorem. Let S be a metrically connected set in a metric space R. Then if we define the distance

$$\rho_s(x,y) = \inf_{\widehat{xy} \subset S} s(\widehat{xy}),$$

the metric thus introduced in S proves to be intrinsic in the above sense. Accordingly we speak of the intrinsic metric induced in S by the metric of the surrounding space R. The definition of the metric of a surface given above is a particular instance of this general definition. Thus, our theorem being applied, we see that this metric is intrinsic in itself.

An M.B.C. must be a surface considered from the intrinsic point of view. Therefore it is natural that our first postulate in the definition of an M.B.C. should be the following one. An M.B.C. is a two-dimensional metric manifold with a metric intrinsic in itself.

1.2. The angle. In order to formulate the condition of the boundedness of the curvature by means of the excesses of triangles we have to define a triangle and an angle. (These definitions will be valid for any metric space.) We define, first, the shortest line or segment xy as a curve joining the points x, y and having the length equal to the distance $\rho(x,y)$ between them. Then it is evident what is understood by a triangle or a polygon. We note that in any manifold and even in any locally compact space with an intrinsic metric each point has a neighbourhood any two points of which can be joined by a segment.

The definition of an angle is given as follows. Let L, M be two curves with the common initial point O. Take the variable points $X \in L$, $Y \in M$ $(X, Y \neq O)$ and construct the plane triangle $O'X'Y'$ with sides equal to the distances OX, OY, XY. Let $\gamma(XY)$ be the angle of this triangle at the vertex O'. We define the upper angle between the curves L and M as

$$\overline{\alpha}(LM) = \overline{\lim_{X,Y\to O}}\, \gamma(XY).$$

This angle always exists.

Further, we say that there exists a definite angle between the curves L, M provided that the limit of the angle $\gamma(XY)$ exists, and in this case we define the angle

$$\alpha(LM) = \lim_{X,Y\to O} \gamma(XY).$$

Making use of the upper angle, which always exists, we define the excess of a triangle T as

$$\delta(T) = \bar{\alpha}+\bar{\beta}+\bar{\gamma}-\pi,$$

$\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ being the upper angles between the sides of T.

1.3. The condition of the boundedness of the curvature. Now we are ready to formulate the second and last postulate which defines an M.B.C. Each point has a neighbourhood U such that the sum of the excesses of pairwise non-overlapping triangles lying in U is uniformly bounded above:

$$\Sigma\,\delta(T) < N.$$

N does not depend upon the set of the triangles and depends upon the neighbourhood U only.

Thus, briefly speaking, an M.B.C. is a 2-manifold with an intrinsic metric and with uniformly bounded sums of the excesses of non-overlapping triangles, at least in certain neighbourhoods which cover the manifold. Sometimes we speak of a metric of bounded curvature, which is preferable in the case where we have to consider not only one but many metrics given in the same manifold, i.e. when the manifolds are topologically mapped on to one and the same manifold.

1.4. Curvature. The definition of curvature is quite natural and runs as follows. We define the positive and the negative parts of the curvature of an open set G as the upper and the lower bounds of the sums of the excesses of the pairwise non-overlapping triangles lying in G

$$\omega^+(G) = \sup\Sigma\,\delta(T), \quad \omega^-(G) = \inf\Sigma\,\delta(T).$$

The curvature itself is defined as

$$\omega(G) = \omega^+(G)+\omega^-(G),$$

and the absolute curvature

$$\Omega(G) = \omega^+-\omega^-.$$

After that one can prolong these set-functions on to the ring of Borel sets by following the routine of measure theory. Then the fundamental fact is that these set-functions prove to be totally additive.

Our conditions concerning the excesses of triangles seem to be, in a certain respect, the minimum one has to suppose in order to have the possibility of defining the curvature as a totally additive set-function.

Zalgaller has given an abstract construction of a measure (non-negative totally additive set-function) which covers the definition of Lebesgue's measure, the above definition of the curvature and of many other set-functions which occur in geometry provided the definition starts from certain magnitudes ascribed to such elementary sets as the area of rectangles in the case of Lebesgue's measure or the excesses of triangles in the case of curvature.

1.5. Some other concepts. We define, further, such concepts as the area of a domain, the direction of a curve at a point, and the integral geodesic curvature (the bend) of a curve. For example, two curves are said to have the same definite direction at their common initial point provided the upper angle between them is equal to zero. The angle between two curves depends upon their directions only, i.e. it is the same for all pairs of curves with the same directions. The set of directions at a given point is isometric, with respect to its angular metric, to the set of generators of a cone.

2. The study of M.B.C. by means of approximation

2.1. With the exception of direct methods the first and most fruitful method in the theory of M.B.C. is that of approximation of general M.B.C. by means of polyhedra. First of all we have the following fundamental theorem: Let an intrinsic metric ρ given in a manifold M be the limit of a uniformly convergent sequence of metrics ρ_n with uniformly bounded positive parts of curvatures. Then ρ is a metric of bounded curvature also, and the curvatures of the metrics ρ_n weakly converge to the curvature ω of ρ in the sense that for any continuous function $f(x)$ different from zero on a compact set only,

$$\int f(x)\,\omega_n(dM) \to \int f(x)\,\omega(dM).$$

In particular, the limit of Riemannian metrics with uniformly bounded positive parts of curvature, i.e. $\int_{K>0} K\,dS$, is a metric of bounded curvature.

2.2. The simplest M.B.C. are manifolds with polyhedral metrics, or, in short, polyhedra. A polyhedron is such a manifold with an intrinsic metric, each point of which has a neighbourhood isometric to a cone. This descriptive definition is equivalent to a constructive one: a polyhedron is a manifold with intrinsic metric constructed of plane triangles, or in other words it allows of a subdivision into triangles isometric to plane ones. The curvature of a polyhedron is concentrated in its vertices, i.e. in such points the neighbourhoods of which do not reduce to pieces of the plane. The whole angle θ around such a point is different from 2π and is connected with the curvature ω of the point by the equation

$$\omega = 2\pi - \theta.$$

The curvature of a polyhedron is the sum of the curvatures at the vertices and its positive part is the sum extended over the vertices with the whole angle $\theta < 2\pi$. The convergence theorem above implies that the limit of polyhedral metrics with uniformly bounded positive parts of curvatures is a metric of bounded curvature.

The converse theorem exists in the following form: Any metric of bounded curvature is a limit of a sequence of polyhedral metrics with uniformly bounded absolute curvatures. Or in a more exact form, let P be a compact polygon in an M.B.C., R, and let ρ be the intrinsic metric induced in P by the metric of R. There exists a sequence of polyhedra P_n with uniformly bounded absolute curvatures, and of mappings of these on to P, such that the metrics determined in P by these mappings uniformly converge to ρ. And the positive and the negative parts of their curvatures weakly converge to the corresponding parts of the curvature of the metric ρ. We say that the convergence is regular.

2.3. If we combine this result with the previous convergence theorem, we get a new definition of an M.B.C. as a manifold which is, at least locally, the limit of polyhedra with uniformly bounded positive parts of curvatures. Polyhedra being, obviously, the limits of Riemannian manifolds, an M.B.C. proves to be the limit of Riemannian manifolds with uniformly bounded positive parts of their integral curvatures. In other words, the class of M.B.C. is the closure of the class of Riemannian or of polyhedral manifolds in the sense of uniform convergence of metrics under the condition of the uniform boundedness of positive parts of curvatures, or, what proves to be the same, the boundedness of absolute integral curvatures.

2.4. The above theorems provide the foundations of a method of studying M.B.C. by means of approximation by polyhedra. This method is applied to the study of some fundamental properties of M.B.C. For instance, we define the area of a polygon P in an M.B.C. as the limit of the areas of polyhedra regularly convergent to P.

In order to ensure a standard application of this method we have to supply ourselves with a number of general theorems concerning the convergence of various magnitudes associated with an M.B.C. and figures in it, such as polygons, curves, angles, area, integral geodesic curvature, etc. In fact, we have such theorems at our disposal.

Suppose, now, we are given a problem concerning M.B.C. Then we formulate it for polyhedra, and attempt to solve it for them. Polyhedra being the objects of elementary geometry, the problem reduces to one of a rather intuitive character. And if we succeed in solving the problem for polyhedra it only remains for us to apply suitable convergence theorems in order to obtain the general result. Most of the concrete results in the theory of M.B.C. have been obtained in this way.

3. Analytic characterization of M.B.C.

3.1. Probably the most important result obtained by means of this method is the following theorem by Reshetnak (1953). The metric of each M.B.C. may be determined by means of a line-element of the form

$$ds^2 = \lambda(dx^2 + dy^2), \tag{1}$$

where $\log\lambda$ is the difference of two subharmonic functions, and conversely, any metric determined by such a line-element, with the same condition for λ, is a metric of bounded curvature.

More exactly the first part of the theorem may be expressed as follows: Let P be a polygon in an M.B.C., R, homeomorphic to a circle. Then, by means of a map of P on to a domain D of the xy-plane, one can introduce in P co-ordinates x, y in such a way that the length of any curve in P which is the image of a broken line L in D is equal to

$$s = \int_L \sqrt{\{\lambda(dx^2 + dy^2)\}}. \tag{2}$$

And if we put $z = x + iy$, $\lambda(x, y) = \lambda(z)$ is representable by means of the following formula

$$\log\lambda(z) = -\frac{1}{\pi}\int_D \log|z-\zeta|\,\omega(dE_\zeta) + h(z), \tag{3}$$

where $\omega(E_\zeta)$ is the curvature of the set in P corresponding to $E_\zeta \subset D$; the integral is understood in the Lebesgue–Radon sense, and $h(z)$ is a suitably chosen harmonic function in D. Since $\omega = \omega^+ + \omega^-$, $\omega^+ \geqslant 0$, $\omega^- \leqslant 0$, the well-known integral representation of subharmonic functions implies that $\log\lambda$ is the difference of two such functions.

This theorem generalizes the well-known fact that the line-element of a regular surface may always be represented in the conformal form (1) and λ is connected with Gaussian curvature by the formula

$$\Delta\log\lambda = -2K\lambda. \tag{4}$$

If we consider this formula as a Poisson equation for $\log \lambda$ we just get the solution in the form (3) with $\omega(dE_\zeta) = K\lambda\, d\xi\, d\eta$.

3.2. Reshetnak's theorem adds to the two above definitions of an M.B.C. (i.e. the initial axiomatic one and the constructive one) a third one, the analytic definition. It opens the way for applications of analytic methods to the study of M.B.C. But so far nobody has followed this way to any considerable extent. Almost all results in the theory so far have been obtained by means of geometric methods.

4. Geometrical methods and some results of the theory of M.B.C.

4.1. There are two geometric methods in the theory of M.B.C., that of approximation by polyhedra and the other one which I call the method of cutting and gluing. It is based upon 'the theorem of gluing'. As a polyhedron is constructed or glued up of triangles, so, more generally, an M.B.C. may be constructed of pieces of given M.B.C., for example, of polygons cut out of any M.B.C., by means of gluing them together along segments of their boundaries. The possibility of such a construction under certain conditions imposed upon the boundaries of the glued pieces is ensured by a theorem which I call the 'theorem of gluing'.

In the simplest case when the glued pieces are polygons the theorem reduces to the following statement: Suppose we are given a complex of polygons cut out of some M.B.C.; suppose the complex is a manifold R with a boundary (possibly void) and that the identified segments of the sides of the polygons have equal lengths. Then if we define for any two points $x, y \in R$ the distance as the greatest lower bound of the lengths of curves joining x, y (the lengths being defined by metrics which are already given in each polygon) then R turns out to be an M.B.C.

In the case of more general domains than polygons an additional condition is necessary. It concerns the integral geodesic curvatures of the boundaries, for these curvatures as segment functions should be of bounded variation. For instance the conditions are fulfilled provided we have pieces of regular surfaces bounded by curves with piecewise continuous geodesic curvature.

4.2. The method of cutting and gluing is as old as geometry itself. The ancient proofs of Pythagoras's theorem as well as many other ancient proofs in elementary geometry consist just in cutting certain figures into suitable pieces and rearranging, or, let me say, gluing those pieces together so as to make the statement obvious. We apply just the same idea to our far more general and abstract figures.