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Localization in Periodic Potentials

From Schrödinger Operators to the Gross–Pitaevskii Equation

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To Anna and my children:
Albert and Edward, Marta and Polina

Contents

Preface

Bose–Einstein condensation was predicted by S.N. Bose and Albert Einstein in 1925: for a gas of non-interacting particles, below a certain temperature there is a phase transition to a localized (condensed) state of lowest energy. This phenomenon was realized experimentally in 1995 in alkali gases by E. Cornell and C. Wieman in Boulder as well as by W. Ketterle at MIT, who all shared the Nobel Prize in 2001. Since that time, the attention of many mathematicians has turned to the analysis of the mean-field model of this phenomenon, which is known as the Gross–Pitaevskii equation or the nonlinear Schrödinger equation with an external potential.

Various trapping mechanisms of Bose–Einstein condensation were realized experimentally, including a parabolic magnetic confinement and a periodic optical lattice. This book is about the Gross–Pitaevskii equation with a *periodic* potential, in particular about the localized modes supported by the periodic potential. The book is written for young researchers in applied mathematics and so it has the main emphasis on the *mathematical properties* of the Gross–Pitaevskii equation. It can nevertheless serve as a reference for theoretical physicists interested in the phenomenon of localization in periodic potentials.

Compared to recent work by Lieb *et al* on the justification of the Gross–Pitaevskii equation as the mean-field approximation of the linear N-body Schrödinger equation [131], this book takes the Gross–Pitaevskii equation as the starting point of analysis. Hence the validity of the mean-field approximation is not questioned when existence and stability of localized modes are considered. On the other hand, the mean-field model is simplified further to the coupled nonlinear Schrödinger equations, the nonlinear Dirac equations, and the discrete nonlinear Schrödinger equations, which are all justified in the framework of the Gross–Pitaevskii equation with a periodic potential.

Besides optical trapping lattices in the context of Bose–Einstein condensation, periodic structures are very common in many other physical problems, including photonic crystals, arrays of coupled optical waveguides, and optically induced photonic lattices. One of the important features of such systems is the existence of *band gaps* in the wave transmission spectra, which support stationary localized modes known as the *gap solitons* or *discrete breathers*. Similar to other solitons, these localized modes realize a balance between periodicity, dispersion, and nonlinearity of the physical system.

Part of this monograph was used for a one-semester graduate course at McMaster University in 2009–2010. The author is grateful to graduate students of the course, particularly to D. Avalo and A. Sakovich, for thoughtful reading of the text and useful corrections. Many parts of this text would not have been possible without contributions of the coauthors on individual projects, in particular I. Barashenkov, M. Chugunova, S. Cuccagna, G. Iooss, P. Kevrekidis, T. Melvin, G. Schneider, A. Stefanov, V. Vougalter, and J. Yang. The author is extremely grateful to all these researchers for their enthusiasm, support, and many fruitful ideas. Numerical computations of localized modes were performed with the assistance of J. Brown during his B.Sc. project in 2009–2010.

Dmitry Pelinovsky