Section 1

Mathematical principles

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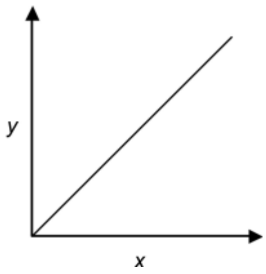
2nd Edition
Excerpt
More Information

Mathematical relationships

Mathematical relationships tend not to be tested as stand-alone topics but an understanding of them will enable you to answer other topics with more authority.

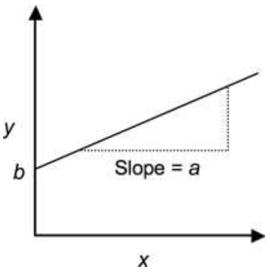
Linear relationships

$y = x$



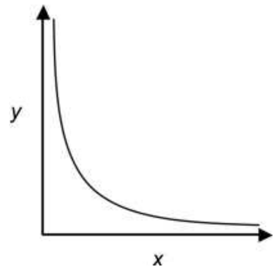
Draw and label the axes as shown. Plot the line so that it passes through the origin (the point at which both x and y are zero) and the value of y is equal to the value of x at every point. The slope when drawn correctly should be at 45° if the scales on both axes are the same.

$y = ax + b$



This line should cross the y axis at a value of b because when x is 0, y must be 0 + b. The slope of the graph is given by the multiplier a. For example, when the equation states that $y = 2x$, then y will be 4 when x is 2, and 8 when x is 4, etc. The slope of the line will, therefore, be twice as steep as that of the line given by $y = 1x$.

Hyperbolic relationships ($y = k/x$)

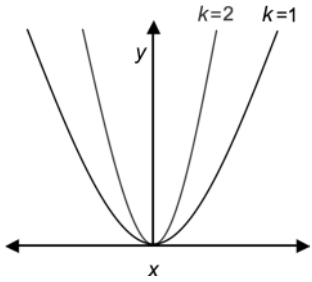


This curve describes any inverse relationship. The commonest value for the constant, *k*, in anaesthetics is 1, which gives rise to a curve known as a rectangular hyperbola. The line never crosses the *x* or the *y* axis and is described as asymptotic to them (see definition below). Boyle's law is a good example (volume = 1/pressure). This curve looks very similar to an exponential decline but they are entirely different in mathematical terms so be sure about which one you are describing.

Asymptote

A curve that continually approaches a given line but does not meet it at any distance.

Parabolic relationships ($y = kx^2$)



These curves describe the relationship $y = x^2$ and so there can be no negative value for *y*. The value for a constant '*k*' alters the slope of the curve in the same way as '*a*' does in the equation $y = ax + b$. The curve crosses the *y* axis at zero unless the equation is written $y = kx^2 + b$, in which case the whole curve is shifted upwards and it crosses at the value of '*b*'.

Exponential relationships and logarithms

Exponential

A condition where the rate of change of a variable at any point in time is proportional to the value of the variable at that time.
or
A function whereby the x variable becomes the exponent of the equation $y = e^x$.

We are normally used to x being represented in equations as the *base* unit (i.e. $y = x^2$). In the exponential function, it becomes the exponent ($y = e^x$), which conveys some very particular properties.

Euler's number

Represents the numerical value 2.71828 and is the base of natural logarithms. Represented by the symbol 'e'.

Logarithms

The power (x) to which a base must be raised in order to produce the number given as for the equation $x = \log_{base}(\text{number})$.

The base can be any number, common numbers are 10, 2 and e (2.71828). $\text{Log}_{10}(100)$ is, therefore, the power to which 10 must be raised to produce the number 100; for $10^2 = 100$, therefore, the answer is $x = 2$. Log_{10} is usually written as log whereas $\log_e$ is usually written ln.

Rules of logarithms

Multiplication becomes addition

$$\log(xy) = \log(x) + \log(y)$$

Division becomes subtraction

$$\log(x/y) = \log(x) - \log(y)$$

Reciprocal becomes negative

$$\log(1/x) = -\log(x)$$

Power becomes multiplication

$\log(x^n) = n.\log(x)$

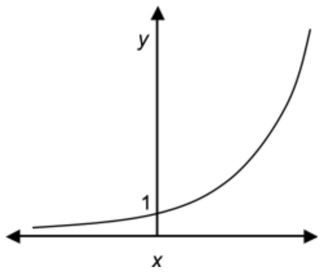
Any log of its own base is one

$\log_{10}(10) = 1$ and $\ln(e) = 1$

Any log of 1 is zero because n^0 always equals 1

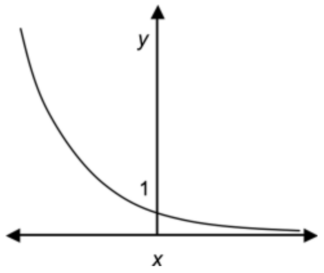
$\log_{10}(1) = 0$ and $\ln(1) = 0$

Basic positive exponential ($y = e^x$)



The curve is asymptotic to the *x* axis. At negative values of *x*, the slope is shallow but the gradient increases sharply when *x* is positive. The curve intercepts the *y* axis at 1 because any number to the power 0 (as in e^0) equals 1. Most importantly, the value of *y* at any point equals the slope of the graph at that point.

Basic negative exponential ($y = e^{-x}$)



The *x* axis is again an asymptote and the line crosses the *y* axis at 1. This time the curve climbs to infinity as *x* becomes more negative. This is because $-x$ is now becoming more positive. The curve is simply a mirror image, around the *y* axis, of the positive exponential curve seen above.

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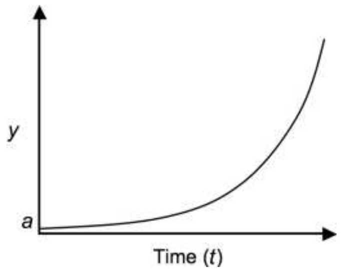
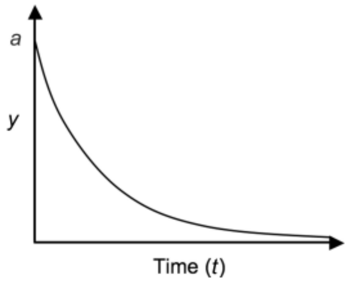
Clinical tear-away positive exponential ($y = a.e^{kt}$)



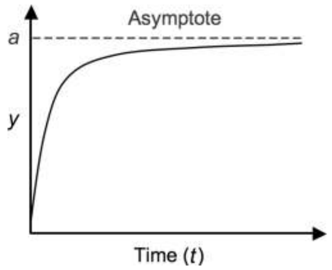
The curve crosses *y* axis at value of *a*. It tends towards infinity as value of *t* increases. This is clearly not a sustainable physiological process but could be seen in the early stages of bacterial replication where *y* equals number of bacteria.

Physiological negative exponential ($y = a.e^{-kt}$)



The curve crosses the *y* axis at a value of *a*. It declines exponentially as *t* increases. The line is asymptotic to the *x* axis. This curve is seen in physiological processes such as drug elimination and lung volume during passive expiration.

Physiological build-up negative exponential ($y = a - b.e^{-kt}$)



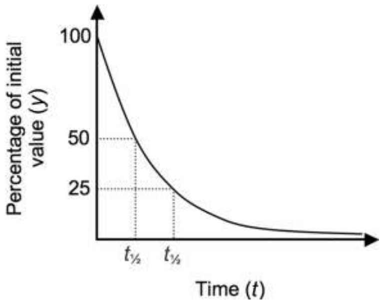
The curve passes through the origin and is asymptotic to a line that would cross the *y* axis at a value of *a*. Although *y* increases with time, the curve is actually a negative exponential. This is because the *rate* of increase in *y* is decreasing exponentially as *t* increases. This curve may be seen clinically as a wash-in curve or that of lung volume during positive pressure ventilation using pressure-controlled ventilation.

Half life

The time taken for the value of an exponential function to decrease by half is the half life and is represented by the symbol $t_{1/2}$
or
the time equivalent of 0.693τ   τ = *time constant*

An exponential process is said to be complete after five half lives. At this point, 96.875% of the process has occurred.

Graphical representation of half life



This curve needs to be drawn accurately in order to demonstrate the principle. After drawing and labelling the axes, mark the key values on the *y* axis as shown. Your curve must pass through each value at an equal time interval on the *x* axis. To ensure this, plot equal time periods on the *x* axis as shown, before drawing the curve. Join the points with a smooth curve that is asymptotic to the *x* axis. This will enable you to describe the nature of an exponential decline accurately as well as to demonstrate easily the meaning of half life.

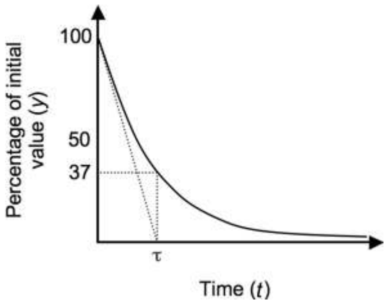
Time constant

The time it would have taken for a negative exponential process to complete, were the initial rate of change to be maintained throughout. Given the symbol τ.
or

The time taken for the value of an exponential to fall to 37% of its previous value.
or
The time taken for the value of an exponential function change by a factor of e^1.
or
The reciprocal of the rate constant.

An exponential process is said to be complete after three time constants. At this point 94.9% of the process has occurred.

Graphical representation of the time constant



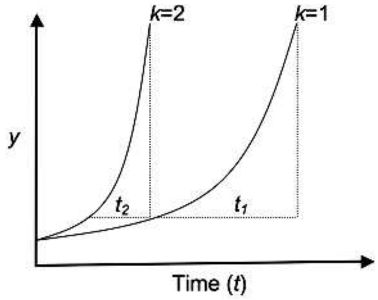
This curve should be a graphical representation of the first and second definitions of the time constant as given above. After drawing and labelling the axes, mark the key points on the *y* axis as shown. Draw a straight line falling from 100 to baseline at a time interval of your choosing. Label this time interval *τ*. Mark a point on the graph where a vertical line from this point crosses 37% on the *y* axis. Finally draw the curve starting as a tangent to your original straight line and falling away smoothly as shown. Make sure it passes through the 37% point accurately. A well-drawn curve will demonstrate the time constant principle clearly.

Rate constant

The reciprocal of the time constant (*k*).
or
A marker of the rate of change of an exponential process.

The rate constant acts as a modifier to the exponent as in the equation $y = e^{kt}$ (e.g. in a savings account, *k* would be the interest rate; as *k* increases, more money is earned in the same period of time and the exponential curve is steeper).

Graphical representation of k ($y = e^{kt}$)



$k = 1$ Draw a standard exponential tear-away curve. To move from $y = e^t$ to $y = e^{t+1}$ takes time t_1.

$k = 2$ This curve should be twice as steep as the first as 'k' acts as a 2 × multiplier to the exponent 't'. As 'k' has doubled, for the same change in y the time taken has halved and this can be shown as t_2 where t_2 is half the value of t_1. The values t_1 and t_2 are also the time constants for the equation because they are, by definition, the reciprocal of the rate constant.

Transforming to a straight line graph

Start with the general equation as follows

$$y = e^{kt}$$

take natural logarithms of both sides

$$\ln y = \ln(e^{kt})$$

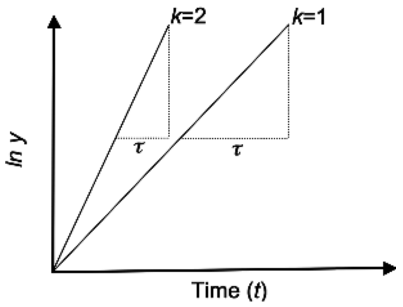
power functions become multipliers when taking logs, giving

$$\ln y = kt.\ln(e)$$

the natural log of e is 1, giving

$$\ln y = kt.1 \text{ or } \ln y = kt$$

You may be expected to perform this simple transformation, or at least to describe the maths behind it, as it demonstrates how logarithmic transformation can make the interpretation of exponential curves much easier by allowing them to be plotted as straight lines $\ln y = kt$.



k = 1 Draw a curve passing through the origin and rising as a straight line at approximately 45°.

k = 2 Draw a curve passing through the origin and rising twice as steeply as the $k = 1$ line. The time constant is half that for the $k = 1$ line.