Chapter I

INTRODUCTION

The subject of the forms of relative equilibrium of a rotating mass of homogeneous gravitating liquid had its inception with the discussion by Newton (1687) of the figure of the Earth. In this it was simply assumed that a possible figure of the free surface would be that of an oblate spheroid with its least axis coincident with the axis of rotation, and it was not until Clairaut’s work many years later that the validity of this postulate was examined. In the first instance Clairaut gave a proof resting on an approximate expression for the potential of a spheroid, but meanwhile Maclaurin (1740) produced an accurate demonstration of the possibility of the spheroidal form, and this led Clairaut also to publish an exact solution in place of his former one. It was rigorously shown by these authors that a spheroid is a possible equilibrium form whatever its eccentricity of meridian section provided it possesses an appropriate quantity of angular momentum.

That an ellipsoid with three unequal axes, the least coinciding with the axis of rotation, is also a possible form of relative equilibrium, provided the angular momentum is greater than a certain amount, remained undiscovered until Jacobi (1834) pointed it out in a letter to the French Academy. Jacobi himself does not appear to have published the result, and it seems first to have been referred to publicly by Poisson shortly after Jacobi’s communication to the Academy. There is perhaps something of an element of surprise about Jacobi’s result in view of the symmetry that might be expected to be associated with any form produced by a rotational field, and the fact also that the Jacobian figures exist only if the angular momentum exceeds a certain amount no doubt contributed to the series being overlooked for so long.

The first member of this Jacobian series is also a Maclaurin spheroid, but thereafter, for greater angular momentum, the equatorial axes are always different, and the elongation of the figure continually increases with the angular momentum. The limiting final form on this series, as infinite angular momentum is approached, has infinite longest axis, while the axis of intermediate length tends to equality with the third and least axis, both of them approaching zero.

Up to this stage researches had not gone beyond the question of the existence of possible equilibrium forms, nor did they do so until the problem was taken up by Poincaré (1885) in the paper that has since become celebrated. His investigation developed a method for studying the difficult question of the stability of the spheroidal and ellipsoidal forms, and necessarily involved also the consideration of the existence and properties of other equilibrium forms. There seems little doubt that a thorough discussion of these questions can be made only by using ellipsoidal harmonic analysis, which by the time Poincaré commenced his work had already been extensively developed by Lamé and others, and lay ready to hand.
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As is well known, where rotating systems are concerned, the stability of steady configurations of relative equilibrium is a more complicated question than for statical systems, and in fact two different kinds of stability may be involved, usually designated by the terms ‘secular’ and ‘ordinary’. The former contemplates the presence within the system of friction (vanishing with the relative velocities), whereas the latter is independent of dissipative action. Poincaré was able to show that if the mass of liquid is regarded as passing slowly along the Maclaurin series in the direction of increasing angular momentum, it becomes secularly unstable at a certain degree of flattening. This particular configuration, called a ‘form of bifurcation’, coincides with the beginning of the Jacobi series. The instability enters for a deformation of the surface involving a certain second order harmonic function. Moreover the Jacobi series can be regarded in its initial stages as the result of making a second-order harmonic deformation of the critical member of the Maclaurin series. This second-order deformation may be pictured as the addition of a stationary wave on the surface of the spheroid. This wave has two places of greatest elevation and two of least elevation on the equator of the spheroid, while the displacement vanishes at the poles, and it has the effect of transforming the spheroid into an ellipsoid with unequal axes. Thus the Jacobi series may be regarded as branching off the Maclaurin series and coming into existence as a result of a surface deformation involving precisely the harmonic terms through which the Maclaurin series has become unstable.

For greater values of the angular momentum along the Maclaurin series, whose members are all possible equilibrium forms, Poincaré showed that further separate points of onset of secular instability occur involving the higher order harmonic deformations. The proof of these results for the spheroids depends for the most part on properties of the ordinary zonal and tesseral harmonic functions to which the ellipsoidal harmonics reduce, in terms of suitable coordinates, when the ellipsoid has two axes equal. The corresponding investigation for the series of Jacobi forms involves the general ellipsoidal harmonics, but in a closely analogous way Poincaré was able to show that the Jacobi figures first become secularly unstable for a certain third-order harmonic deformation, and that for greater elongations and angular momentum, further separate configurations of instability occur, corresponding successively to harmonics of fourth-order, fifth-order, and so on.

The occurrence of these instabilities accordingly suggests the possibility that at each point of bifurcation there branches off a new series of configurations of equilibrium given initially in each case by the addition of a surface deformation expressed by the same harmonic through which the instability first appears. As the Jacobi series is described in the direction of increasing angular momentum, instability first enters for a certain third-order harmonic, and the initial forms of the branch series will thus be obtainable by adding a surface deformation of this kind. The surface deformation involved in this case has three places of greatest elevation and three of least elevation in the equatorial plane, and vanishes at all points in the plane orthogonal to this defined by the two shorter axes. The amplitudes of the deformations supposed added are assumed infinitesimal
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throughout, but if a small finite amplitude is adopted for the purposes of illustrating the nature of the new equilibrium forms, the resulting surface (Fig. 16, p. 110) was considered by Poincaré to bear marked resemblance to that of a pear. Accordingly the initial figures of the series became known as the ‘pear-shaped’ figures, or sometimes the ‘piriform’ figures. The difficult problem that accordingly next arose was to ascertain whether or not the pear-shaped series is initially stable or unstable. This in essence is Tchebychev’s problem, and it gave rise to extensive investigations by Poincaré, Darwin, Liapounoff, and Jeans, to mention the chief authors. Minor contributions on various technical points connected with it were made by Schwarzschild, H. F. Baker, and others.

The properties of the fluid mass having been assigned, as uniform, gravitating, and if necessary viscous, the general problem of possible equilibrium forms and their stability can be stated as a purely theoretical one, but Poincaré to some extent, and Darwin almost entirely, were interested in the problem from its possible cosmonogonical applications. The general form of the pear-shaped figure undoubtedly gave rise to the notion that if the mass were stable and evolved by equilibrium forms along this series, with the furrow continually deepening as the figure elongated, the final result would be two detached masses rotating in circular orbital motion about each other. Thus it seemed plain to Darwin that the dynamical theory, if it could be established in accordance with these ideas, would give a strong theoretical basis for supposing this to be the method of genesis of double systems in the celestial universe. Indeed, Darwin eventually announced that he had proved the pear-shaped series to be initially stable and hence that this would in fact be the course of development.

Now in order to establish that the pear-shaped figures are stable it is sufficient to show that they are secularly stable. On the other hand, if the series were shown to be secularly unstable, it would in general require further investigation to decide in what manner the system is next likely to develop. At all events, Darwin approached the problem by a method that aimed only at settling the question of secular stability, as also did Liapounoff, though this author appears to have been attracted solely by the theoretical problem, and in no wise interested in the astronomical implications of the results, which pass unmentioned in his numerous papers. Subsequently, Jeans also concerned himself with establishing the secular stability or otherwise, apparently under the impression that ordinary stability was not relevant to the problem.

Had these writers been able finally to conclude that the pear-shaped figure was secularly stable, their treatments would have represented a complete solution of the immediate problem (though there would have still remained the question of how far the pear-shaped series continued to be stable with increasing angular momentum, just as this was necessary for the Maclaurin and Jacobi series). But in point of fact Liapounoff, and later Jeans, concluded that the figure was quite certainly secularly unstable, and moreover Jeans claimed that Darwin’s original investigation, when properly interpreted with certain technical errors set right, itself in reality led to this same result. As far as the objective of Liapounoff’s inquiry was concerned he had attained it, but Jeans’s expressed
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intention was to obtain theoretical evidence of the course of dynamical stellar evolution, albeit under what have since proved perhaps unduly idealized conditions. The desired information, however, is not conclusively provided by the knowledge that the system is secularly unstable, for this means that the pear-shaped form itself never comes into existence, and if the continuation of the Jacobi series remained ordinarily stable, as might possibly happen, the rate of departure from the critical Jacobi figure might not take place at all rapidly if frictional effects were small. For example, the lunar orbit is secularly unstable, but ordinarily stable, and its evolution under frictional forces proceeds extremely slowly and would cease altogether if friction were absent.

Thus to complete the information that may be derived from the consideration of small deformations of the system it is necessary to determine whether the Jacobi series remains ordinarily stable or not beyond the critical Jacobi figure. It is automatically so before this stage is reached as a consequence of its secular stability. This question requires a totally different treatment from that of secular stability, since it is necessary for its solution to study the actual periods of possible small oscillations of the system, and not merely the manner in which some single entity, such as the moment of momentum, changes along the initial stages of the pear-shaped series. The determination of the ordinary stability of the Jacobi series has been undertaken and solved by Cartan, who has succeeded in proving that for displacements involving the third-order harmonic through which the ellipsoids first become secularly unstable, they simultaneously become ordinarily unstable.

With this information available the nature of the development beyond the critical Jacobi figure can be studied with more certainty. For if the free surface receives a displacement involving third-order harmonics, and any general physical disturbance may be assumed to contain such terms in its expression, the amplitudes will begin to increase exponentially at a rate independent of the amount of friction present. The system can no longer oscillate about an equilibrium form, since none exists of a stable character, and instead a dynamical motion must ensue until the system succeeds in finding its way to a new state of steady motion. The equations of small motion of the system permit this development to be followed only so long as the velocities and displacements involved in it remain small, but with increasing amplitude the approximations leading to linear equations of motion become less accurate. The system must, however, reach eventually some other steady condition involving no further dissipation of energy, and the interesting question arises as to what this final configuration will be. Unfortunately it is not possible to investigate the question in detail by anything approaching rigorous means, but it may well be, as has always been maintained, that the result will be a division of the original mass into two detached portions. However, if this view is correct, there is necessarily an important difference from Darwin’s ideas on the course of development, for it can be shown not only that the pieces must be of considerably different sizes, but what is more important still, that they must separate to infinity. The final steady state would then consist of two separate unequal stable masses receding
with constant relative velocity, the original excess angular momentum causing
the instability reappearing now as orbital angular momentum.

From the point of view of cosmogony the main question of interest is to obtain
as rigorous a demonstration of this course of development as possible. It would
also be of interest to give as complete an account as possible of the whole evolu-
tion of the problem, but the literature of the subject is so extensive and much
of it of an exploratory character, that it would scarcely be practicable in a single
volume to give more than an outline of it. Yet it has seemed worthwhile to
give the full mathematical discussion of such parts as are essential to establish-
ing the extreme plausibility of the course of evolution described above. To do this
we begin by discussing the subject of stability with particular reference to
rotating systems. This is followed by a discussion of the spherical, spheroidal,
and ellipsoidal forms, together with certain of their properties that can be
established by simple means, as illustrations of the dynamical theory. Next is
developed the ellipsoidal harmonic analysis required for further progress, when
the necessary properties of Lamé’s functions are derived, and then using this
mathematical technique an account is given of Poincaré’s investigation of the
secular stability of the Maclaurin and Jacobi series. This is followed by an
account of Cartan’s discussion of the ordinary stability of the Jacobi forms.
Finally, the subsequent development of the system is considered and its possible
cosmogonical implications discussed.
Chapter II

STABILITY

STABILITY OF STATICAL SYSTEMS

Equilibrium configurations

Let us consider a mechanical system whose position can be specified by $n$ generalized coordinates $q_1, q_2, ..., q_n$, and its motion at any instant by the generalized velocities $\dot{q}_1, \dot{q}_2, ..., \dot{q}_n$, where the dots denote differentiation with respect to time $t$. Also let us suppose that the forces acting on the system are derivable from a potential energy function $V$ depending on the coordinates only, so that

$$V = V(q_i).$$

The kinetic energy $T$ will in general depend on both the coordinates and the velocities, and will be a homogeneous quadratic function of the latter. Thus

$$T = T(q_i, \dot{q}_i).$$

Motion of the system will take place in accordance with the Lagrangian equations

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} = - \frac{\partial V}{\partial q_i} \quad (i = 1, 2, ..., n),$$

which possess the energy integral

$$T + V = \text{constant}. \quad (2)$$

Hence the possible equilibrium configurations, within the limits of the number of coordinates adopted, are determined by the $n$ equations

$$\frac{\partial V}{\partial q_i} = 0 \quad (i = 1, 2, ..., n), \quad (3)$$

which are simply the conditions that $V$ is stationary.

These equations may have certain admissible solutions, each of which will correspond to a possible equilibrium state, and we may denote a particular one by

$$q_i = a_i \quad (i = 1, 2, ..., n). \quad (4)$$

Moreover, it can readily be seen that if $V$ is an absolute minimum in the configuration then the equilibrium is stable. For, in accordance with the energy integral (2), if a slight disturbance from equilibrium occurs, it follows, since $T$ cannot become negative, that $V$ cannot increase above its equilibrium value by more than a very small amount. This means in turn that none of the coordinates can deviate by more than a small amount from its equilibrium value, and therefore that during the motion the system must always remain in the immediate neighbourhood of the equilibrium configuration, which is what is meant by stability.
Linear series of configurations

If the description of the system contains a parameter \( \mu \), say, not itself dependent on any of the generalized coordinates, which for any reason is slowly changing, or is so regarded, the potential \( V \) will in general depend on \( \mu \), and the values of \( q_i \) giving the solutions of (3) will also depend on \( \mu \). Thus the particular solution (4) may be written

\[
q_i = a_i(\mu) \quad (i = 1, 2, \ldots, n),
\]

where the \( a_i \)'s are now functions of \( \mu \).

When \( \mu \) undergoes a small change \( d\mu \), the solution will become

\[
q_i = a_i(\mu + d\mu) = a_i + \frac{da_i}{d\mu} d\mu,
\]

and hence the original configuration will give place to an adjacent configuration for the new system with slightly different \( \mu \). Thus starting from a given equilibrium position, a continuous series of other possible configurations is obtained as \( \mu \) slowly varies. Such a set of configurations Poincaré has termed a ‘linear series’. The essence of the idea is the presence within the system of a changing parameter which, however, varies sufficiently gradually for the change not to affect equilibrium at any stage.

Stability

Let us consider next what may happen to the stability of the system as it moves slowly along a linear series. We examine this question first by diagrammatic means. If we take \((q_i; \mu)\) as rectangular coordinates with the \( \mu \)-axis vertical, then for different values of \( V \) on the right-hand side, the relation

\[
V(q_i; \mu) = V
\]

will correspond in two dimensions to a family of curves, or in three dimensions to surfaces. With more than two coordinates, \( q_i \), the relation will represent hyper-surfaces according to the number of dimensions involved. As there is only one value of \( V \) for a given point \((q_i; \mu)\) these surfaces are essentially non-intersecting.

The condition that the tangent plane to a given \( V \)-surface shall be perpendicular to the \( \mu \)-axis is simply that \( dV = 0 \) for \( d\mu = 0 \), which is equivalent to the relations (3). The equilibrium configurations therefore correspond to the points where the \( V \)-surfaces are horizontal. In Fig. 1, let \( V_1 \) and \( V_2 \) represent two surfaces of the family and suppose \( V_2 > V_1 \). Then the point \( P_1 \) represents an equilibrium configuration and the corresponding value of \( \mu \) is \( \mu_1 \), as shown. The heavy line joining successive equilibrium points \( P_1, P_2, \ldots \) represents the linear series.

Slight displacements of the system, without change of \( \mu \), will be represented, for \( \mu = \mu_1 \) say, by points such as \( P_1' \) in the tangent plane at \( P_1 \). In the circumstances postulated, such changes will involve an increase of potential energy if the \( V \)-surfaces are concave downwards as shown, since the general direction of \( V \) increasing is as indicated. Thus the condition for stability can be stated in the form that the \( V \)-surfaces must be concave downwards.
Clearly, if the general direction of increasing $V$ were in the present case the other way, stability would require the surfaces to be concave upwards. The rule is therefore that for stability the concavities must be towards the direction of $V$ decreasing.

Exchange of stabilities

As the representative point of the configuration passes along the linear series no new feature arises so long as the $V$-surfaces remain concave in the same direction. However, it may happen that they gradually develop in one of the ways shown in the diagrams of Fig. 2, which illustrate some of the possible forms that the $V$-surfaces may take. The analytical discussion given later covers all possible cases. At present we consider these four examples.

(i) As the linear series is described in the direction indicated by the arrow, the concavities eventually become the other way and additional downward concavities appear. Accordingly, at the critical point $C$ another linear series $BCB$ cuts across the original series $A_1C A_4$. Such a point is termed a ‘point of bifurcation’, and the corresponding equilibrium position a ‘form of bifurcation’. If the series $A_1C$ was at first stable, the change in direction of the concavities means that beyond $C$ the continuation $C A_4$ of this series must be unstable. That is, at $C$ the original series loses its stability. On the other hand, in the case depicted, the new series $BCB$ has its concavities the same way as the series $A_1C$, and hence the configurations of this series will be stable. There has occurred a transfer of stability to the members of the new series. For each value of $\mu$ corresponding to the portion $A_1C$, there exists only one equilibrium form, and this is stable, but for each value beyond this there are three possible forms of which two are stable and one unstable.

(ii) Here, assuming the initial series $A_1C$ to be stable, the continuation $C A_4$ of this series must again be unstable, but the series $BCB$ must now also be unstable. In this case there is therefore a disappearance of stability at $C$.

(iii) Here two configurations of equilibrium are possible to begin with for each value of $\mu$, and if that on the series $A_1C$ is stable, that on $C A_4$ is unstable. Not only is stability lost at $C$, but for values of $\mu$ beyond that of $C$ there are no equilibrium forms possible at all.
(iv) Here only one configuration is possible for values of \(\mu\) corresponding to the series \(A_1C\), and again there are no equilibrium forms possible beyond that corresponding to \(C\).

Cases (i) and (ii) are of special importance in the actual problems with which we shall be concerned. In the application of these considerations to physical systems it is convenient always to choose for \(\mu\) some parameter that increases, or decreases, monotonically as the system gradually evolves. We shall see that one or other of such quantities as the angular velocity or angular momentum can usually be postulated to satisfy this requirement. Assuming such a choice to have been made, it is seen that in (ii) the members of the unstable series \(BCB\) cannot actually come into existence, even though the value of \(\mu\) would permit it theoretically, if the system is originally in a stable configuration on \(A_1C\). It could, of course, if necessary, be regarded as placed in one of these unstable positions.
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It may be remarked that the term ‘exchange of stabilities’, which is often used in connexion with cases such as (i) and (ii), is to some extent a misleading one, because stability is not necessarily transferred to the new series. There is always a loss of stability of the original series, but the new series, if one exists, may or may not be stable.

Following Jeans, these possible cases can be represented schematically by the following idealized diagrams:

\begin{align*}
\text{(i)} & \quad \text{(ii)} \\
\text{(iii)} & \quad \text{(iv)}
\end{align*}

Fig. 3. The heavy lines denote stable series and the dotted lines unstable series. The arrows indicate the direction of evolution.

\textbf{Rules for graphical determination of stability}

These considerations therefore lead to the following rules for determining diagrammatically how the stability of a system changes at a point of bifurcation:

If in evolving along a stable series represented by a vertical line a point of bifurcation is reached, stability will be lost by the first series and transferred to the new series if the curve representing the latter turns upwards. This is case (i) above.

If the new series turns downwards, as in (ii), configurations on it cannot come into existence, and the continuation of the original series is unstable.

If the original series itself turns downwards, as in (iii), or ceases altogether, as in (iv), there are no further equilibrium forms either stable or unstable for values of the parameter immediately beyond that corresponding to the point of bifurcation.

It is highly important to understand that the number of coordinates used to describe the system must remain the same throughout for these results to be valid. Continuous systems, such as liquid masses, require for their complete description an infinite number of coordinates, but by restricting the mass to special forms the number of coordinates required can be made finite. Thus, if only ellipsoidal forms are admitted, then but two coordinates are needed, for if \(a, b, c\) denote the semi-axes these must always be related by \(abc = \text{constant}\). If only spheroidal forms are admitted, then \(a = b\) also, and one coordinate suffices. If a diagram of the above kind is drawn for the Maclaurin series no point of bifurcation appears because no other equilibrium series is possible within these restrictions. On the other hand, if a diagram is drawn for the ellipsoidal series, it is found to have a point of bifurcation at the point where the Maclaurin series crosses it.

It is for the present reason that the possibility of the pear-shaped figure only became demonstrated when the full number of freedoms could be coped with