The Mathematics of Signal Processing

Arising from courses taught by the authors, this largely self-contained treatment is ideal for mathematicians who are interested in applications or for students from applied fields who want to understand the mathematics behind their subject.

Early chapters cover Fourier analysis, functional analysis, probability and linear algebra, all of which have been chosen to prepare the reader for the applications to come. The book includes rigorous proofs of core results in compressive sensing and wavelet convergence. Fundamental is the treatment of the linear system \( y = \Phi x \) in both finite and infinite dimensions. There are three possibilities: the system is determined, overdetermined or underdetermined, each with different aspects.

The authors assume only basic familiarity with advanced calculus, linear algebra and matrix theory, and modest familiarity with signal processing, so the book is accessible to students from the advanced undergraduate level. Many exercises are also included.

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The Mathematics of Signal Processing

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Preface

 Basically, this is a book about mathematics, pitched at the advanced undergraduate/beginning graduate level, where ideas from signal processing are used to motivate much of the material, and applications of the theory to signal processing are featured. It is meant for math students who are interested in potential applications of mathematical structures and for students from the fields of application who want to understand the mathematical foundations of their subject. The first few chapters cover rather standard material in Fourier analysis, functional analysis, probability theory and linear algebra, but the topics are carefully chosen to prepare the student for the more technical applications to come. The mathematical core is the treatment of the linear system $y = \Phi x$ in both finite-dimensional and infinite-dimensional cases. This breaks up naturally into three categories in which the system is determined, overdetermined or underdetermined. Each has different mathematical aspects and leads to different types of application. There are a number of books with some overlap in coverage with this volume, e.g., [11, 15, 17, 19, 53, 69, 71, 72, 73, 82, 84, 95, 99, 101], and we have profited from them. However, our text has a number of features, including its coverage of subject matter, that together make it unique. An important aspect of this book on the interface between fields is that it is largely self-contained. Many such books continually refer the reader elsewhere for essential background material. We have tried to avoid this. We assume the reader has a basic familiarity with advanced calculus and with linear algebra and matrix theory up through the diagonalization of symmetric or self-adjoint matrices. Most of the remaining development of topics is self-contained. When we do need to call on technical results not proved in the text, we try to be specific. Little in the way of formal knowledge about signal processing is assumed. Thus while
this means that many interesting topics cannot be covered in a text of modest size, the topics that are treated have a logical coherence, and the reader is not continually distracted by appeals to other books and papers. There are many exercises. In most of the volume the logic of the mathematical topics predominates, but in a few chapters, particularly for compressive sensing and for parsimonious representation of data, the issues in the area of application predominate and mathematical topics are introduced as appropriate to tackle the applied problems. Some of the sections, designated by “Digging deeper” are more technical and can be mostly skipped on a first reading. We usually give a nontechnical description of the principal results of these sections. The book is sufficiently flexible to provide relatively easy access to new ideas for students or instructors who wish to skip around, while filling in the background details for others. We include a large list of references for the reader who wants to “dig deeper.” In particular, this is the case in the chapter on the parsimonious representation of data.

This book arose from courses we have both taught and from ongoing research. The idea of writing the book originated while the first author was a New Directions Professor of Imaging at the Institute for Mathematics and its Applications, The University of Minnesota during the 05–06 academic year. The authors acknowledge support from the National Science Foundation; the Centre for High Performance Computing, Cape Town; the Institute for Mathematics and its Applications, University of Minnesota; the School of Computational and Applied Mathematics, the University of the Witwatersrand, Johannesburg; Georgia Southern University; and the United States Office of Airforce Research. We are indebted to a large number of colleagues and students who have provided valuable feedback on this project, particularly Li Lin and Peter Mueller who tested the compressive sensing algorithms. All figures in this book were generated by us from open source programs such as CVX, Maple or MATLAB, or from licensed MATLAB wavelet and signal processing toolboxes.

In closing, we thank the staff at Cambridge University Press, especially David Tranah and Jon Billam, for their support and cooperation during the preparation of this volume and we look forward to working with them on future projects.