The Mathematics of Signal Processing

Arising from courses taught by the authors, this largely self-contained treatment is ideal for mathematicians who are interested in applications or for students from applied fields who want to understand the mathematics behind their subject.

Early chapters cover Fourier analysis, functional analysis, probability and linear algebra, all of which have been chosen to prepare the reader for the applications to come. The book includes rigorous proofs of core results in compressive sensing and wavelet convergence. Fundamental is the treatment of the linear system $y = \Phi x$ in both finite and infinite dimensions. There are three possibilities: the system is determined, overdetermined or underdetermined, each with different aspects.

The authors assume only basic familiarity with advanced calculus, linear algebra and matrix theory, and modest familiarity with signal processing, so the book is accessible to students from the advanced undergraduate level. Many exercises are also included.

STEVEN B. DAMELIN is Full Professor of Mathematics and Director for the Unit of Advances in Mathematics and its Applications, USA.

WILLARD MILLER, JR. is Professor Emeritus in the School of Mathematics at the University of Minnesota.

Cambridge Texts in Applied Mathematics

All titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing, visit www.cambridge.org/mathematics.

Nonlinear Dispersive Waves
MARK J. ABLOWITZ

Complex Variables: Introduction and Applications (2nd Edition)
MARK J. ABLOWITZ & ATHANASSIOS S. FOKAS

Scaling
G. I. R. BARENBLATT

Introduction to Symmetry Analysis
BRIAN J. CANTWELL

Hydrodynamic Instabilities
FRANÇOIS CHARRU

A First Course in Continuum Mechanics
OSCAR GONZALEZ & ANDREW M. STUART

Theory of Vortex Sound
M. S. HOWE

Applied Solid Mechanics
PETER HOWELL, GREGORY KOZYREFF & JOHN OCKENDON

Practical Applied Mathematics: Modelling, Analysis, Approximation
SAM HOWISON

A First Course in the Numerical Analysis of Differential Equations (2nd Edition)
ARIEH ISERLES

A First Course in Combinatorial Optimization
JON LEE

An Introduction to Parallel and Vector Scientific Computation
RONALD W. SHONKWILER & LEW LEFTON

The Mathematics of Signal Processing

STEVEN B. DAMELIN
Unit of Advances in Mathematics and its Applications

WILLARD MILLER, JR.
University of Minnesota

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town,
Singapore, São Paulo, Delhi, Mexico City

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9781107013223

First published 2012

Printed in the United Kingdom at the University Press, Cambridge

A catalog record for this publication is available from the British Library

ISBN 978-1-107-01322-3 Hardback
ISBN 978-1-107-60104-8 Paperback

Dedicated to our parents, children and partners

Cambridge University Press
978-1-107-60104-8 - The Mathematics of Signal Processing
Steven B. Damelin and Willard Miller
Frontmatter
More information

Contents

	Preface		*page* xi
	Introduction		1
1	**Normed vector spaces**		3
	1.1	Definitions	4
	1.2	Inner products and norms	10
	1.3	Finite-dimensional ℓ_p spaces	14
	1.4	Digging deeper: completion of inner product spaces	20
	1.5	Hilbert spaces, L_2 and ℓ_2	25
	1.6	Orthogonal projections, Gram–Schmidt orthogonalization	39
	1.7	Linear operators and matrices, LS approximations	46
	1.8	Additional exercises	64
2	**Analytic tools**		73
	2.1	Improper integrals	73
	2.2	The gamma functions and beta functions	78
	2.3	The sinc function and its improper relatives	79
	2.4	Infinite products	83
	2.5	Additional exercises	86
3	**Fourier series**		92
	3.1	Definitions, real Fourier series and complex Fourier series	92
	3.2	Examples	96
	3.3	Intervals of varying length, odd and even functions	97
	3.4	Convergence results	99
	3.5	More on pointwise convergence, Gibbs phenomena	107
	3.6	Further properties of Fourier series	113

3.7 Digging deeper: arithmetic summability and Fejér's theorem 116
3.8 Additional exercises 123

4 The Fourier transform 127
4.1 Fourier transforms as integrals 127
4.2 The transform as a limit of Fourier series 129
4.3 L_2 convergence of the Fourier transform 135
4.4 The Riemann–Lebesgue lemma and pointwise convergence 140
4.5 Relations between Fourier series and integrals: sampling 146
4.6 Fourier series and Fourier integrals: periodization 152
4.7 The Fourier integral and the uncertainty principle 154
4.8 Digging deeper 157
4.9 Additional exercises 161

5 Compressive sampling 164
5.1 Introduction 164
5.2 Algebraic theory of compressive sampling 168
5.3 Analytic theory of compressive sampling 172
5.4 Probabilistic theory of compressive sampling 183
5.5 Discussion and practical implementation 201
5.6 Additional exercises 206

6 Discrete transforms 208
6.1 Z transforms 208
6.2 Inverse Z transforms 211
6.3 Difference equations 213
6.4 Discrete Fourier transform and relations to Fourier series 214
6.5 Fast Fourier transform (FFT) 222
6.6 Approximation to the Fourier transform 223
6.7 Additional exercises 224

7 Linear filters 230
7.1 Discrete linear filters 230
7.2 Continuous filters 233
7.3 Discrete filters in the frequency domain 235
7.4 Other operations on discrete signals 238
7.5 Additional exercises 240

8 Windowed Fourier and continuous wavelet transforms. Frames 242
8.1 The windowed Fourier transform 243

	8.2	Bases and frames, windowed frames	251
	8.3	Affine frames	268
	8.4	Additional exercises	270
9	**Multiresolution analysis**		272
	9.1	Haar wavelets	272
	9.2	The multiresolution structure	284
	9.3	Filter banks and reconstruction of signals	296
	9.4	The unitary two-channel filter bank system	304
	9.5	A perfect reconstruction filter bank with $N = 1$	306
	9.6	Perfect reconstruction for two-channel filter banks	307
	9.7	Halfband filters and spectral factorization	309
	9.8	Maxflat filters	312
	9.9	Low pass iteration and the cascade algorithm	317
	9.10	Scaling functions by recursion: dyadic points	320
	9.11	The cascade algorithm in the frequency domain	329
	9.12	Some technical results	332
	9.13	Additional exercises	335
10	**Discrete wavelet theory**		341
	10.1	L_2 convergence	345
	10.2	Accuracy of approximation	354
	10.3	Smoothness of scaling functions and wavelets	359
	10.4	Additional exercises	365
11	**Biorthogonal filters and wavelets**		367
	11.1	Resumé of basic facts on biorthogonal filters	367
	11.2	Biorthogonal wavelets: multiresolution structure	370
	11.3	Splines	382
	11.4	Generalizations of filter banks and wavelets	390
	11.5	Finite length signals	395
	11.6	Circulant matrices	397
	11.7	Additional exercises	400
12	**Parsimonious representation of data**		401
	12.1	The nature of digital images	402
	12.2	Pattern recognition and clustering	418
	12.3	Image representation of data	426
	12.4	Image compression	429
	12.5	Additional exercises	433
	References		437
	Index		443

Preface

Basically, this is a book about mathematics, pitched at the advanced undergraduate/beginning graduate level, where ideas from signal processing are used to motivate much of the material, and applications of the theory to signal processing are featured. It is meant for math students who are interested in potential applications of mathematical structures and for students from the fields of application who want to understand the mathematical foundations of their subject. The first few chapters cover rather standard material in Fourier analysis, functional analysis, probability theory and linear algebra, but the topics are carefully chosen to prepare the student for the more technical applications to come. The mathematical core is the treatment of the linear system $y = \Phi x$ in both finite-dimensional and infinite-dimensional cases. This breaks up naturally into three categories in which the system is determined, overdetermined or underdetermined. Each has different mathematical aspects and leads to different types of application. There are a number of books with some overlap in coverage with this volume, e.g., [11, 15, 17, 19, 53, 69, 71, 72, 73, 82, 84, 95, 99, 101], and we have profited from them. However, our text has a number of features, including its coverage of subject matter, that together make it unique. An important aspect of this book on the interface between fields is that it is largely self-contained. Many such books continually refer the reader elsewhere for essential background material. We have tried to avoid this. We assume the reader has a basic familiarity with advanced calculus and with linear algebra and matrix theory up through the diagonalization of symmetric or self-adjoint matrices. Most of the remaining development of topics is self-contained. When we do need to call on technical results not proved in the text, we try to be specific. Little in the way of formal knowledge about signal processing is assumed. Thus while this means that many interesting topics cannot be covered in a text of modest size, the topics that are treated have a logical coherence, and the reader is not continually distracted by appeals to other books and papers. There are many exercises. In most of the volume the logic of the mathematical topics predominates, but in a few chapters, particularly for compressive sensing and for parsimonious representation of data, the issues in the area of application predominate and mathematical topics are introduced as appropriate to tackle the applied problems. Some of the sections, designated by "Digging deeper" are more technical and can be mostly skipped on a first reading. We usually give a nontechnical description of the principal results of these sections. The book is sufficiently flexible to provide relatively easy access to new ideas for students or instructors who wish to skip around, while filling in the background details for others. We include a large list of references for the reader who wants to "dig deeper." In particular, this is the case in the chapter on the parsimonious representation of data.

This book arose from courses we have both taught and from ongoing research. The idea of writing the book originated while the first author was a New Directions Professor of Imaging at the Institute for Mathematics and its Applications, The University of Minnesota during the 05–06 academic year. The authors acknowledge support from the National Science Foundation; the Centre for High Performance Computing, Cape Town; the Institute for Mathematics and its Applications, University of Minnesota; the School of Computational and Applied Mathematics, the University of the Witwatersrand, Johannesburg; Georgia Southern University; and the United States Office of Airforce Research. We are indebted to a large number of colleagues and students who have provided valuable feedback on this project, particularly Li Lin and Peter Mueller who tested the compressive sensing algorithms. All figures in this book were generated by us from open source programs such as CVX, Maple or MATLAB, or from licensed MATLAB wavelet and signal processing toolboxes.

In closing, we thank the staff at Cambridge University Press, especially David Tranah and Jon Billam, for their support and cooperation during the preparation of this volume and we look forward to working with them on future projects.