Quadratic functions

Exercise 1B

4  
\( a \) \( y = x^2 - 6x + 11 \)
\( = (x-3)^2 - 9 + 11 \)
\( = (x-3)^2 + 2 \)

b Minimum value of \( y \) is 2.

5  
\( a \) Minimum at \((3,6)\) \( \Rightarrow y = a(x-3)^2 + 6 \)
So \( a = -3 \), \( b = 6 \)

b Curve passes through \((1,14)\), so substituting \( x = 1 \) and \( y = 14 \) into \( y = a(x-3)^2 + 6 \):
\( 14 = a(1-3)^2 + 6 \)
\( 8 = 4a \)
\( a = 2 \)

6  
\( a \) \( 2x^2 + 4x - 1 = 2 \left[ x^2 + 2x \right] - 1 \)
\( = 2 \left[ (x+1)^2 - 1 \right] - 1 \)
\( = 2(x+1)^2 - 2 - 1 \)
\( = 2(x+1)^2 - 3 \)

b Line of symmetry is \( x = -1 \)

c \( 2x^2 + 4x - 1 = 0 \)
\( 2(x+1)^2 - 3 = 0 \)
\( 2(x+1)^2 = 3 \)
\( (x+1)^2 = \frac{3}{2} \)
\( x + 1 = \pm \sqrt{\frac{3}{2}} \)
\( x = -1 \pm \sqrt{\frac{3}{2}} \)

Exercise 1C

4  
\( a \) \( 2x^2 + 5x - 12 = (2x-3)(x+4) \)

b The graph crosses the \( x \)-axis where \( y = 0 \), i.e. where \( 2x^2 + 5x - 12 = 0 \):
\( 2x^2 + 5x - 12 = 0 \)
\( (2x-3)(x+4) = 0 \)
\( 2x - 3 = 0 \) or \( x = 4 = 0 \)
\( x = \frac{3}{2} \) or \( x = -4 \)

So the intersections with the \( x \)-axis are at \( \left( \frac{3}{2}, 0 \right) \) and \( (-4, 0) \).

5  
Roots at \(-5\) and \(2\)
\( \Rightarrow y = a(x - 2)(x+5) \)
\( = ax^2 + 3ax - 10a \)
So \( c = -10a \) and \( b = 3a \).

\( y \)-intercept at \(3 \Rightarrow c = 3 \)
\( \therefore 3 = -10a \)
\( a = -\frac{3}{10} \)

and \( b = 3 \left( -\frac{3}{10} \right) = -\frac{9}{10} \)

Exercise 1D

5  
\( 3x^2 + 4x + 1 \Leftrightarrow 3x^2 - 4x - 1 = 0 \)
Using the quadratic formula with \( a = 3 \), \( b = -4 \) and \( c = -1 \):
\[
x = \frac{(-4) \pm \sqrt{(-4)^2 - 4 \times 3 \times (-1)}}{2 \times 3}
\]
\[
= \frac{4 \pm \sqrt{28}}{6}
\]
\[
= \frac{2 \pm \sqrt{7}}{3}
\]

The discriminant with \(a = 4\), \(b = 1\) and \(c = \frac{1}{16}\) is
\[
\Delta = b^2 - 4ac
\]
\[
= 1^2 - 4 \times 4 \times \frac{1}{16}
\]
\[
= 0
\]
So there is only one root, i.e. the vertex lies on the x-axis.

Equal roots when discriminant is zero:
\[
\Delta = b^2 - 4ac = 0
\]
\[
(-4)^2 - 4 \times 4 \times 2m = 0
\]
\[
16 - 8m^2 = 0
\]
\[
m^2 = 2
\]
\[
m = \pm \sqrt{2}
\]

Tangent to the x-axis implies equal roots, so discriminant is zero:
\[
\Delta = b^2 - 4ac = 0
\]
\[
(2k + 1)^2 - 4 \times (-3) \times (-4k) = 0
\]
\[
4k^2 + 4k + 1 - 48k = 0
\]
\[
4k^2 - 44k + 1 = 0
\]
\[
k = \frac{44 \pm \sqrt{44^2 - 4 \times 4 \times 1}}{2 \times 4}
\]
\[
= \frac{44 \pm \sqrt{1936}}{8}
\]
\[
= \frac{11 \pm \sqrt{30}}{2}
\]

No real solutions when discriminant \(\Delta < 0\):
\[
b^2 - 4ac < 0
\]
\[
(-6)^2 - 4 \times 1 \times 2k < 0
\]
\[
36 - 8k < 0
\]
\[
k > \frac{9}{2}
\]

For a quadratic to be non-negative (\(\geq 0\)) for all \(x\), it must have at most one root, so \(\Delta \leq 0\) and \(a > 0\).
\[
b^2 - 4ac \leq 0
\]
\[
(-3)^2 - 4 \times 2 \times (2c - 1) \leq 0
\]
\[
9 - 16c + 8 \leq 0
\]
\[
c \geq \frac{17}{16}
\]

**COMMENT**

Note that \(\Delta \leq 0\) is not sufficient in general for a quadratic to be non-negative. The condition \(a > 0\) is also necessary to ensure that the quadratic has a positive shape (opening upward) rather than a negative shape (opening downward), so that the curve remains above the x-axis and never goes below it, as would be the case if \(a < 0\). In this question \(a\) was given as positive \(2\), so we did not need to use this condition at all.

For a quadratic to be negative for all \(x\), it must have no real roots, so \(\Delta < 0\) and \(a < 0\).
\[
b^2 - 4ac < 0
\]
\[
3^2 - 4 \times m \times (-4) < 0
\]
\[
9 + 16m < 0
\]
\[
m < -\frac{9}{16}
\]
COMMENT
The condition \( \alpha < 0 \) ensures that the function is negative shaped and therefore remains below the x-axis. In this case \( \alpha = m \), and it followed from the condition on \( \Delta \) that \( \alpha < 0 \), as seen in the answer.

12
The two zeros of \( ax^2 + bx + c \) are 
\[
\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}.
\]
The positive difference between these zeros is
\[
\left| \frac{-b + \sqrt{b^2 - 4ac}}{2a} - \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right| = \frac{2\sqrt{b^2 - 4ac}}{2a} = \frac{\sqrt{b^2 - 4ac}}{a}.
\]
So, in this case,
\[
k^2 - 12 = 69
\]
\( k^2 = 81 \)
\( k = \pm 9 \)

COMMENT
Note that modulus signs were used in the general expression for the positive distance, as \( \alpha \) could be negative. Here \( \alpha = 1 \) and so the modulus was not required in the specific case in this question.

Exercise 1E
3
\[
y = x^2 - 4 \quad \text{...(1)}
\]
\[
y = 8 - x \quad \text{...(2)}
\]

Substituting (1) into (2):
\[
x^2 - 4 = 8 - x
\]
\[
x^2 + x - 12 = 0
\]
\[
(x - 3)(x + 4) = 0
\]
\( x = 3 \) or \( x = -4 \)

Substituting into (2):
\( x = 3: \quad y = 8 - 3 = 5 \)
\( x = -4: \quad y = 8 - (-4) = 12 \)
So the points of intersection are (3, 5) and (-4, 12).

4
\[
y = 2x^2 - 3x + 2 \quad \text{...(1)}
\]
\[
3x + 2y = 5 \quad \text{...(2)}
\]
Substituting (1) into (2):
\[
x = -\frac{1}{4} \quad \text{and} \quad x = 1
\]

Substituting into (1):
\[
x = -\frac{1}{4}: \quad y = 2 \left( \frac{1}{4} \right)^2 - 3 \left( \frac{1}{4} \right) + 2 = \frac{23}{8}
\]
\( x = 1: \quad y = 2x^2 - 3x + 2 = 1 \)
So the solutions are \( \left( -\frac{1}{4}, \frac{23}{8} \right) \) and (1, 1).

5a
\[
x^2 - 6x + y^2 - 2y - 8 = 0 \quad \text{...(1)}
\]
\[
y = x - 8 \quad \text{...(2)}
\]
Substituting (2) into (1):
\[
x^2 - 6x + (x - 8)^2 - 2(x - 8) - 8 = 0
\]
\[
2x^2 - 24x + 72 = 0
\]
\[
x^2 - 12x + 36 = 0
\]
as required.

b
\[
x^2 - 12x + 36 = 0
\]
\( (x - 6)^2 = 0 \)
\( x = 6 \)
There is only one point of intersection, which means that the line is tangent to the circle.

6 \[ y = mx + 3 \quad \ldots (1) \]
\[ y = 3x^2 - x + 5 \quad \ldots (2) \]
Substituting (1) into (2):
\[ mx + 3 = 3x^2 - x + 5 \]
\[ 3x^2 - x - mx + 2 = 0 \]
\[ 3x^2 - (m + 1)x + 2 = 0 \]
Only one intersection means that this quadratic has a single root. So \( \Delta = 0 \):
\[ b^2 - 4ac = 0 \]
\[ [- (m + 1)]^2 - 4 \times 3 \times 2 = 0 \]
\[ (m + 1)^2 = 24 \]
\[ m + 1 = \pm \sqrt{24} \]
\[ m = -1 \pm \sqrt{24} = -1 \pm 2\sqrt{6} \]

Exercise 1F

1 Let one number be \( x \) and the other be \( y \).

Sum of \( x \) and \( y \) is 8: \( x + y = 8 \quad \ldots (1) \)

Product is 9.75: \( xy = 9.75 \quad \ldots (2) \)

From (1): \( y = 8 - x \quad \ldots (3) \)
Substituting (3) into (2):
\[ x(8 - x) = 9.75 \]
\[ x^2 - 8x + 9.75 = 0 \]
\[ 4x^2 - 32x + 39 = 0 \]
\[ (2x - 3)(2x - 13) = 0 \]
\[ x = \frac{3}{2} \quad \text{or} \quad x = \frac{13}{2} \]
The two numbers are 1.5 and 6.5.

2 The length is \( x \); let the width be \( y \).

Perimeter of 12:
\[ 2(x + y) = 12 \]
\[ x + y = 6 \]
\[ y = 6 - x \]
Area \( A = xy \)
\[ = x(6 - x) \]
\[ = 6x - x^2 \]
Completing the square:
\[ A = -(x^2 - 6x) \]
\[ = -(x - 3)^2 + 9 \]

So the maximum area is 9 cm\(^2\) (and occurs when \( x = y = 3 \), i.e., when the rectangle is a square with side 3 cm).

Note that the negative sign makes the quadratic negative shaped, which results in a maximum rather than minimum turning point.

3 a New fencing required is 200 m, so
\[ 2x - 10 + y = 200 \]
\[ \Rightarrow y = 210 - 2x \]
Area \( A = xy \)
\[ = x(210 - 2x) \]
\[ = 210x - 2x^2 \]
b Completing the square:

\[ A = -2 \left[ x^2 - 105x \right] \]
\[ = -2 \left( x - \frac{105}{2} \right)^2 - \left( \frac{105}{2} \right)^2 \]
\[ = 2 \left( x - \frac{105}{2} \right)^2 - \frac{105^2}{2} \]

\[ \therefore \text{maximum area when } x = \frac{105}{2} = 52.5 \text{m, for which } y = 210 - 2 \times \frac{105}{2} = 105 \text{m} \]

4 a Ball is at ground level when \( h = 0 \).

\[ 8t - 4.9t^2 = 0 \]
\[ t(8 - 4.9t) = 0 \]
\[ t = 0 \text{ or } t = \frac{8}{4.9} = 1.63 \text{ (3SF)} \]

So the ball returns to the ground after 1.63 s.

b By symmetry of a quadratic, maximum height (vertex) is halfway between the roots, \( t = 0 \) and \( t = \frac{8}{4.9} \).

\[ t_{\text{max}} = \frac{0 + \frac{8}{4.9}}{2} = \frac{4}{4.9} \]

\[ h_{\text{max}} = 8 \left( \frac{4}{4.9} \right) - 4.9 \left( \frac{4}{4.9} \right)^2 \]
\[ = \frac{32 - 16}{4.9} = \frac{16}{4.9} = 3.27 \text{ m (3SF)} \]

5 a Perimeter of 60:

\[ 2x + \frac{1}{2} \pi y = 60 \]
\[ \Rightarrow 2x = 60 - \frac{\pi y}{2} \]
\[ \Rightarrow x = 30 - \frac{\pi y}{4} \]
\[ A = xy + \frac{1}{2} \pi \left( \frac{y}{2} \right)^2 \]
\[ = \left( 30 - \frac{\pi y}{4} \right) y + \frac{\pi y^2}{8} \]
\[ = 30y - \frac{\pi y^2}{4} + \frac{\pi y^2}{8} \]
\[ = 30y - \frac{1}{8} \pi y^2 \]

b Finding the roots of the area expression:

\[ \left( 30y - \frac{1}{8} \pi y^2 \right) = 0 \]
\[ y \left( 30 - \frac{1}{8} \pi y \right) = 0 \]
\[ y = 0 \text{ or } y = \frac{240}{\pi} \]

By the symmetry of a quadratic, the maximum area (vertex) is halfway between the roots:

\[ \frac{0 + \frac{240}{\pi}}{2} = \frac{120}{\pi} \]

When \( y = \frac{120}{\pi} \),

\[ x = 30 - \frac{\pi y}{4} \]
\[ = 30 - \frac{\pi}{4} \left( \frac{120}{\pi} \right) \]
\[ = 30 - 30 \]
\[ = 0 \]
This could also have been solved using a GDC or by completing the square.

Mixed examination practice 1

Short questions

1. (a) \(x^2 + 5x - 14 = (x + 7)(x - 2)\)
   (b) \(x^2 + 5x - 14 = 0\)
      \((x + 7)(x - 2) = 0\)
      \(x = -7\) \text{ or } \(x = 2\)

2. (a) Positive quadratic so the vertex is a minimum point
   (b) Minimum at \((3, 7) \Rightarrow y = (x - 3)^2 + 7\)
      So \(a = 3, b = 7\)

3. Maximum \(y\)-value is 48 \(\Rightarrow c = 48\).
   Passes through \((-2, 0)\) and \((6, 0)\) means that its roots are \(x = -2\) and \(x = 6\). The line of symmetry is midway between the roots: i.e. at \(x = 2\), so \(b = 2\).
   Substituting \(x = -2\) and \(y = 0\) into 
   \[ y = a(x - 2)^2 + 48: \]
   \[ 0 = a(-2 - 2)^2 + 48 \]
   \[ 0 = 16a + 48 \]
   \[ a = -3 \]
   So \(a = -3, b = 2\) and \(c = 48\).

4. Roots at \(x = k\) and \(x = k + 4 \Rightarrow\) line of symmetry is \(x = k + 2\) (midway between the roots).
   So the \(x\)-coordinate of the turning point is \(k + 2\).

5. (a) Roots at \(-\frac{1}{2}\) and 2, so
   \[ f(x) = \left(x + \frac{1}{2}\right)(x - 2) \]
   i.e. \(p = -\frac{1}{2}, q = 2\)

Finding the roots of the total profit function:

4\(n(50 - n) = 0\)
\(n = 0\) \text{ or } \(n = 50\)

By the symmetry of a quadratic, the maximum lies halfway between the roots, i.e. at \(n = 25\).

COMMENT

This could also have been solved by completing the square.
b Line of symmetry is midway between
the roots: \( x = \frac{2 + \left( \frac{-1}{2} \right)}{2} = \frac{3}{4} \)
\( \therefore \) x-coordinate of C is \( \frac{3}{4} \)

6 • Negative quadratic \( \Rightarrow a \) is negative
• Negative y-intercept \( \Rightarrow c \) is negative
• Single (repeated) root \( \Rightarrow b^2 - 4ac = 0 \)
• Line of symmetry \( x = \frac{b}{2a} \) is positive
\( \Rightarrow b \) is positive (as \( a \) is negative)

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<td><strong>Expression</strong></td>
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<td>( c )</td>
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<td>( b^2 - 4ac )</td>
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<td>( b )</td>
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7 a \( x^2 - 10x + 35 = (x-5)^2 - 25 + 35 \\
\quad = (x-5)^2 + 10 \)

b From (a), the minimum value of \( x^2 - 10x + 35 \) is 10.

Hence the maximum value of
\[ \frac{1}{(x^2 - 10x + 35)} = \frac{1}{10^3} = \frac{1}{1000} \]

8 Equal roots \( \Rightarrow \Delta = 0 \)
\( b^2 - 4ac = 0 \)
\( (k+1)^2 - 4 \times 2k \times 1 = 0 \)
\( k^2 - 6k + 1 = 0 \)
\( k = \frac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2} \)

9 No real roots \( \Rightarrow \Delta < 0 \)
\( b^2 - 4ac < 0 \)
\( 6^2 - 4 \times 2 \times k < 0 \)
\( 36 - 8k < 0 \)
\( k > \frac{9}{2} \)

10 Only one zero \( \Rightarrow \Delta = 0 \)
\( b^2 - 4ac = 0 \)
\( [- (k+1)]^2 - 4 \times 1 \times 3 = 0 \)
\( (k+1)^2 - 12 = 0 \)
\( k+1 = \pm 2\sqrt{3} \)
\( k = -1 \pm 2\sqrt{3} \)

11 a Roots of \( x^2 - kx + (k-1) = 0 \) are
\[ k \pm \frac{\sqrt{k^2 - 4(k-1)}}{2} = \frac{k \pm \sqrt{k^2 - 4k + 4}}{2} \]
\[ = \frac{k \pm (k-2)^2}{2} \]
\[ = \frac{k \pm (k-2)}{2} \]
\( = k - 1 \) or \( 1 \)

\( \therefore \) \( \alpha = k - 1, \ \beta = 1 \)

b \( \alpha^2 + \beta^2 = 17 \)
\( (k-1)^2 + 1 = 17 \)
\( k^2 - 2k + 2 = 17 \)
\( k^2 - 2k - 15 = 0 \)
\( (k-5)(k+3) = 0 \)
\( k = 5 \) or \( k = -3 \)
Long questions

1 a i Square perimeter = 4x

ii Circle perimeter = 2\pi y

b 4x + 2\pi y = 8 \Rightarrow x = 2 - \frac{\pi}{2} y

c Area of square + area of circle

= x^2 + \pi y^2

= \left(2 - \frac{\pi y}{2}\right)^2 + \pi y^2

= 4 - 2\pi y + \frac{\pi^2 y^2}{4} + \pi y^2

= \frac{\pi}{4}(\pi + 4)y^2 - 2\pi y + 4

d Completing the square:

\[
A = \frac{\pi}{4}(\pi + 4) \left[ y^2 - \frac{8}{\pi + 4}y + \frac{16}{\pi(\pi + 4)} \right]
\]

= \frac{\pi}{4}(\pi + 4) \left[ \left(y - \frac{4}{\pi + 4}\right)^2 - \left(\frac{4}{\pi + 4}\right)^2 + \frac{16}{\pi(\pi + 4)} \right]

So the minimum area occurs when \( y = \frac{4}{\pi + 4} \)

Percentage of wire in circle

\[
\frac{\text{length of wire in circle}}{\text{total length of wire}} \times 100\%
\]

= \frac{2\pi y}{8} \times 100\%

= \frac{2\pi \left(\frac{4}{\pi + 4}\right)}{8} \times 100\%

= 44.0\% (3SF)

COMMENT

Note that it isn’t necessary to simplify the constant in the expression for A after completing the square, as the question asks only for the value of y where the area is minimised and not for the actual value of that minimum.
2 a Car A has position \((2t - 50, 0)\) and Car B has position \((0, 15t - 30)\).

\[
d^2 = \left( x_2 - x_1 \right)^2 + \left( y_2 - y_1 \right)^2
= \left[ 0 - (2t - 50) \right]^2 + \left[ (15t - 30) - 0 \right]^2
= (20t - 50)^2 + (15t - 30)^2
= 400t^2 - 2000t + 2500 + 225t^2 - 900t + 900
= 625t^2 - 2900t + 3400
\]

b Completing the square:

\[
d^2 = 625 \left[ t^2 - \frac{116}{25} t \right] + 3400
= 625 \left[ \left( t - \frac{58}{25} \right)^2 - \left( \frac{58}{25} \right)^2 \right] + 3400
= 625 \left( t - \frac{58}{25} \right)^2 - \frac{3400}{25} + 3400
= 625 \left( t - \frac{58}{25} \right)^2 + 36
\]

So \(d^2 \geq 36\) and, since \(d > 0\), it follows that the minimum value of \(d\) is 6 km.

3 a Vertex on the x-axis

\[\Rightarrow\text{has only one root, so } \Delta = 0.\]

\[b^2 - 4ac = 0\]

\[36 - 4k = 0\]

\[k = 9\]

b Equation of first graph is

\[y = x^2 - 6x + 9 = (x - 3)^2\]

So vertex is at \((3, 0)\).

Second graph has vertex at \((-2, 5)\), so its equation is \(y = a(x + 2)^2 + 5\)

It passes through \((3, 0)\); substituting into the equation gives

\[0 = a(3 + 2)^2 + 5\]

\[25a = -5\]

\[a = -\frac{1}{5}\]

\[\Rightarrow y = -\frac{1}{5}(x + 2)^2 + 5\]

\[= -\frac{1}{5}(x^2 + 4x + 4) + 5\]

\[= -\frac{1}{5}x^2 - \frac{4}{5}x + \frac{21}{5}\]

c For intersection of \(y = x^2 - 6x + 9\) and \(y = -\frac{1}{5}x^2 - \frac{4}{5}x + \frac{21}{5}\)

\[x^2 - 6x + 9 = -\frac{1}{5}x^2 - \frac{4}{5}x + \frac{21}{5}\]

\[5x^2 - 30x + 45 = -x^2 - 4x + 21\]

\[6x^2 - 26x + 24 = 0\]

\[3x^2 - 13x + 12 = 0\]

\[(3x - 4)(x - 3) = 0\]

\[x = \frac{4}{3} \text{ or } x = 3\]

\[x = 3\] is the point of intersection at the vertex \((3, 0)\) of the first graph.

To find the y-coordinate of the other point, substitute \(x = \frac{4}{3}\) into \(y = (x - 3)^2\):

\[y = \left( \frac{4}{3} - 3 \right)^2 = \frac{25}{9}\]

So the other point of intersection is \(\left( \frac{4}{3}, \frac{25}{9} \right)\).