1

Möbius Transformations

In this chapter we provide a brief review of Möbius transformations on n-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 2$). A good reference for these topics is the monograph by A. F. Beardon [11]. First, however, we begin with a review of notation that will be used throughout these notes.

1.1 Notation

For $x, y \in \mathbb{R}^n$ we let $\langle x, y\rangle = \sum_{j=1}^n x_j y_j$ denote the usual inner product on $\mathbb{R}^n$ and $|x| = \sqrt{\langle x, x\rangle}$ the length of the vector x. For $a \in \mathbb{R}^n$ and $r > 0$, the ball $B(a, r)$ and sphere $S(a, r)$ are given respectively by

$$B(a,r) = \{x \in \mathbb{R}^n : |x - a| < r\},$$
$$S(a,r) = \{x \in \mathbb{R}^n : |x - a| = r\}.$$

The unit ball and unit sphere with center at the origin will simply be denoted by $\mathbb{B}$ and $\mathbb{S}$ respectively.[1] The **one-point compactification** of $\mathbb{R}^n$, denoted $\hat{\mathbb{R}}^n$, is obtained by appending the point ∞ to $\mathbb{R}^n$. A subset U of $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ is open if it is an open subset of $\mathbb{R}^n$, or if U is the complement in $\hat{\mathbb{R}}^n$ of a compact subset C of $\mathbb{R}^n$. With this topology $\hat{\mathbb{R}}^n$ is compact.

For a subset D of $\mathbb{R}^n$, $\overline{D}$ denotes the closure of D, Int (D) the interior of D, ∂D the boundary of D, and $\widetilde{D}$ the complement of D in $\mathbb{R}^n$. Also if E and F are sets, $E \setminus F$ denotes the complement of F in E, that is, $E \setminus F = E \cap \widetilde{F}$.

The study of functions of n-variables is simplified with the use of multi-index notation. For an ordered n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$, where each α_j is a non-negative integer, the following notational conventions will be used throughout:

[1] If we wish to emphasize the dimension n, we will use the notation $\mathbb{B}_n$ and $\mathbb{S}_n$ to denote the unit ball and sphere in $\mathbb{R}^n$.

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \quad x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

and

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

If Ω is an open subset of $\mathbb{R}^n$, we denote by $C^k(\Omega)$, $k = 0, 1, 2, \ldots$ the set of real-valued (or complex-valued) functions f on Ω for which $D^\alpha f$ exists and is continuous for all multi-indices α with $|\alpha| \le k$. Thus $C^0(\Omega)$, or simply $C(\Omega)$, denotes the set of real-valued (or complex-valued) continuous functions on Ω, and $C^\infty(\Omega)$ the set of infinitely differentiable functions on Ω. Also, the set of functions $f \in C^k(\Omega)$ for which $D^\alpha f$, $|\alpha| \le k$, has a continuous extension to $\overline{\Omega}$ will be denoted by $C^k(\overline{\Omega})$. If $f : \Omega \mapsto \mathbb{R}$, then the **support** of f, denoted $\operatorname{supp} f$, is defined as

$$\operatorname{supp} f = \overline{\{x \in \Omega : f(x) \neq 0\}}.$$

The set of continuous functions on Ω with compact support will be denoted by $C_c(\Omega)$. The notations $C_c^k(\Omega)$ and $C_c^\infty(\Omega)$ have the obvious meanings.

A linear transformation $A : \mathbb{R}^n \mapsto \mathbb{R}^n$ is said to be **orthogonal** if $|Ax| = |x|$ for all $x \in \mathbb{R}^n$. The set of orthogonal transformations of $\mathbb{R}^n$ will be denoted by $O(n)$. If A is represented by the $n \times n$ matrix $(a_{i,j})$, then A is orthogonal if and only if

$$\sum_{k=1}^n a_{i,k} a_{j,k} = \delta_{i,j} = \begin{cases} 1 & i = j, \\ 0, & i \neq j. \end{cases}$$

If $\psi(x) = (\psi_1(x), \ldots, \psi_n(x))$ is a C^1 mapping of an open subset Ω of $\mathbb{R}^n$ into $\mathbb{R}^n$, then the derivative $\psi'(x)$ is the $n \times n$ matrix given by

$$\psi'(x) = \left(\frac{\partial \psi_i}{\partial x_j} \right)_{i,j=1}^n,$$

and the **Jacobian** J_ψ of the transformation ψ is given by $J_\psi(x) = \det \psi'(x)$.

1.2 Inversion in Spheres and Planes

Definition 1.2.1 *The **inversion**[2] (or reflection) in the sphere $S(a, r)$ is the function $\phi(x)$ defined by*

[2] Although we will mainly be interested in the case $n \ge 2$, the formulas for inversions in spheres and planes are still meaningful when $n = 1$.

$$\phi(x) = a + \left(\frac{r}{|x-a|}\right)^2 (x-a). \tag{1.2.1}$$

The inversion in the unit sphere $\mathbb{S}$ is the mapping $\phi(x) = x^*$ where

$$x^* = \begin{cases} \dfrac{x}{|x|^2} & x \neq 0, \infty, \\ 0 & x = \infty, \\ \infty & x = 0. \end{cases}$$

Thus (1.2.1) can now be rewritten as

$$\phi(x) = a + r^2(x-a)^*.$$

The reflection $\phi(x)$ is not defined at $x = a$. Since $|\phi(x)| \to \infty$ as $x \to a$ we set $\phi(a) = \infty$. Also, since $\lim_{|x|\to\infty} |\phi(x) - a| = 0$, we set $\phi(\infty) = a$. Thus ϕ is defined on all of $\hat{\mathbb{R}}^n$, and it is easily shown that ϕ is continuous in the topology of $\hat{\mathbb{R}}^n$. A straightforward computation also shows that $\phi(\phi(x)) = x$ for all $x \in \hat{\mathbb{R}}^n$. Thus ϕ is a one-to-one continuous map of $\hat{\mathbb{R}}^n$ onto $\hat{\mathbb{R}}^n$ satisfying $\phi(x) = x$ if and only $x \in S(a, r)$.

In addition to reflection in a sphere we also have reflection in a plane. For $a \in \mathbb{R}^n$, $a \neq 0$, and $t \in \mathbb{R}$, the plane $P(a, t)$ is defined by

$$P(a, t) = \{x \in \mathbb{R}^n : \langle x, a\rangle = t\}.$$

By convention ∞ belongs to every plane $P(a, t)$.

Definition 1.2.2 *The* **inversion** *(or reflection) in the plane $P(a, t)$ is the function $\psi(x)$ defined by*

$$\psi(x) = x + \lambda a,$$

where $\lambda \in \mathbb{R}$ is chosen so that $\frac{1}{2}(x + \psi(x)) \in P(a, t)$.

Solving for λ gives

$$\psi(x) = x - 2[\langle x, a\rangle - t]a^*, \quad x \in \mathbb{R}^n. \tag{1.2.2}$$

For the mapping ψ we have

$$|\psi(x)|^2 = |x|^2 + O(|x|),$$

and as a consequence $\lim_{|x|\to\infty} |\psi(x)| = \infty$. Thus as above we define $\psi(\infty) = \infty$. With this definition the mapping ψ again satisfies $\psi(\psi(x)) = x$ for all $x \in \hat{\mathbb{R}}^n$. Thus ψ is a one-to-one continuous map of $\hat{\mathbb{R}}^n$ onto itself with $\psi(x) = x$ if and only if $x \in P(a, t)$. It is well known that each inversion (in a sphere or a plane) is orientation-reversing and conformal (see [11, Theorem 3.1.6]).

1.3 Möbius Transformations

Definition 1.3.1 *A **Möbius transformation** of* $\hat{\mathbb{R}}^n$ *is a finite composition of inversions in spheres or planes.*

Clearly the composition of two Möbius transformations is again a Möbius transformation, as is the inverse of a Möbius transformation. The group of Möbius transformations on $\hat{\mathbb{R}}^n$ is called the **general Möbius group** and is denoted by $\mathrm{GM}(\hat{\mathbb{R}}^n)$. Although not immediately obvious, both translation and magnification by a constant are Möbius transformations. The translation $x \mapsto x + a$, $a \in \mathbb{R}^n$, is the composition of inversion in the plane $\langle x, a\rangle = 0$ followed by inversion in the plane $\langle x, a\rangle = \frac{1}{2}|a|^2$. Likewise, the magnification or scalar multiplication $x \mapsto kx$, $k > 0$, is also a Möbius transformation in that it is the inversion in $\mathbb{S}$ followed by the inversion in $S(0, \sqrt{k})$. Furthermore, every Euclidean isometry of $\mathbb{R}^n$ is a composition of at most $n + 1$ reflections in planes ([11, Theorem 3.1.3]).

We conclude this section by showing that every Möbius transformation maps a sphere or plane onto a sphere or plane. We will use the term "sphere" to denote either a sphere of the form $S(a, r)$ or a plane $P(a, t)$. Since every inversion ψ in a plane $P(a, t)$ can be written as

$$\psi(x) = x + \lambda a,$$

the mapping ψ clearly maps a "sphere" onto a "sphere." To show that an inversion ϕ in a sphere $S(a, r)$ preserves "spheres," it suffices to show that the mapping x^* preserves "spheres."

For any set $E \subset \mathbb{R}^n$, we let $E^* = \{x^* : x \in E\}$. A set $E \subset \mathbb{R}^n$ is a sphere or a plane if and only if

$$E = \{x \in \mathbb{R}^n : b|x|^2 - 2\langle x, a\rangle + c = 0\},$$

where b and c are real and $a \in \mathbb{R}^n$. By convention, ∞ satisfies this equation if and only if $b = 0$, that is, E is a plane. Now it is easily seen that E^* has the same form with the roles of b and c reversed. Finally, it is an easy exercise to show that for $a \in \mathbb{R}^n$ and $r > 0$,

$$S^*(a, r) = \begin{cases} S\left(\dfrac{a}{(|a|^2 - r^2)}, \dfrac{r}{||a|^2 - r^2|}\right) & \text{if } 0 \notin S(a, r), \\ P(a, \tfrac{1}{2}) & \text{if } 0 \in S(a, r). \end{cases} \tag{1.3.1}$$

We conclude this section with one more useful formula that will be required later. If ϕ is inversion in the sphere $S(a, r)$, then a straightforward computation gives

$$|\phi(y) - \phi(x)| = \frac{r^2|y - x|}{|x - a||y - a|}. \tag{1.3.2}$$

For details on the above the reader is referred to [11].

2

Möbius Self-Maps of the Unit Ball

In this chapter we will provide a characterization of the Möbius transformations of $\hat{\mathbb{R}}^n$ mapping the unit ball $\mathbb{B}$ onto $\mathbb{B}$ that is similar to the characterization of the Möbius mappings of the unit disc in $\mathbb{C}$ onto itself.

In the complex plane $\mathbb{C}$, every analytic Möbius transformation ψ mapping the unit disc $\mathbb{D}$ onto itself can be written as $\psi(z) = e^{i\theta}\varphi_w(z)$, where for $w \in \mathbb{D}$,

$$\varphi_w(z) = \frac{w - z}{1 - \overline{w}z}.$$

The mappings $\varphi_w(z)$ satisfy $\varphi_w(0) = w$, $\varphi_w(w) = 0$, and $\varphi_w(\varphi_w(z)) = z$ for all $z \in \mathbb{D}$. Furthermore, the mapping $\varphi_w(z)$ also satisfies

$$1 - |\varphi_w(z)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \overline{w}z|^2}.$$

2.1 Möbius Transformations of $\mathbb{B}$

In this section we define an analogous family of Möbius transformations $\{\varphi_a : a \in \mathbb{B}\}$ mapping $\mathbb{B}$ onto $\mathbb{B}$ having the property that every Möbius transformation ψ mapping $\mathbb{B}$ onto itself can be written as $\psi = A \circ \varphi_a$, where $a \in \mathbb{B}$ and $A \in O(n)$. For $a \in \mathbb{B}$, we first set

$$\psi_a(x) = a + (1 - |a|^2)(a - x)^*. \tag{2.1.1}$$

Since the mapping ψ_a is a composition of Möbius transformations, ψ_a is a Möbius transformation of $\hat{\mathbb{R}}^n$ mapping 0 to a^* and a to ∞. By a straightforward computation we have

$$|\psi_a(x)|^2 = \frac{|a - x|^2 + (1 - |a|^2)(1 - |x|^2)}{|a - x|^2}, \tag{2.1.2}$$

and as a consequence

$$|\psi_a(x)|^2 - 1 = \frac{(1-|a|^2)(1-|x|^2)}{|x-a|^2}. \tag{2.1.3}$$

From the above it follows immediately that ψ_a maps $\mathbb{B}$ onto $\hat{\mathbb{R}}^n \setminus \overline{\mathbb{B}}$.

We now define the mapping φ_a by

$$\varphi_a(x) = \psi_a(x)^* = \frac{\psi_a(x)}{|\psi_a(x)|^2}. \tag{2.1.4}$$

If we set[1]

$$\rho(x,a) = |x-a|^2 + (1-|a|^2)(1-|x|^2) = |a|^2|a^*-x|^2, \tag{2.1.5}$$

then the mapping φ_a can be expressed as

$$\varphi_a(x) = \frac{a|x-a|^2 + (1-|a|^2)(a-x)}{\rho(x,a)}. \tag{2.1.6}$$

As a consequence of (2.1.3)

$$1 - |\varphi_a(x)|^2 = \frac{(1-|x|^2)(1-|a|^2)}{\rho(x,a)}. \tag{2.1.7}$$

Thus φ_a is a Möbius transformation mapping $\mathbb{B}$ onto $\mathbb{B}$ with $\varphi_a(0) = a$ and $\varphi_a(a) = 0$. That φ_a maps $\mathbb{B}$ onto $\mathbb{B}$ follows immediately from the fact that ψ_a maps $\mathbb{B}$ onto $\hat{\mathbb{R}}^n \setminus \overline{\mathbb{B}}$ and that x^* maps $\hat{\mathbb{R}}^n \setminus \overline{\mathbb{B}}$ onto $\mathbb{B}$. We will shortly prove that φ_a also satisfies $\varphi_a(\varphi_a(x)) = x$ for all $x \in \mathbb{B}$. In the unit disk $\mathbb{D}$, for $z, w \in \mathbb{D}$, $\rho(z,w) = |1-\overline{w}z|^2$ and the mappings $\varphi_w(z)$ as defined by (2.1.6) are precisely the functions $(w-z)\big/(1-\overline{w}z)$.

One of the advantages of the mappings φ_a is that the function $(a,x) \mapsto \varphi_a(x)$ is not only continuous on $\overline{\mathbb{B}} \times \overline{\mathbb{B}}$ but also differentiable in each of the variables. At this point we will include several computations involving derivatives of the mappings φ_a that will be required in the proof of Theorem 2.1.2 and also later in the sequel. Let $y_j(x)$ denote the jth coordinate of $y(x) = \varphi_a(x)$. Then by straightforward computations we have

$$\frac{\partial y_j}{\partial x_i}(0) = -\delta_{i,j}(1-|a|^2), \qquad \frac{\partial y_j}{\partial x_i}(a) = \frac{-\delta_{i,j}}{(1-|a|^2)}, \tag{2.1.8}$$

$$\frac{\partial^2 y_j}{\partial x_i^2}(0) = (1-|a|^2)[2a_j - 4a_i\delta_{i,j}]. \tag{2.1.9}$$

Hence

$$\varphi_a'(0) = -(1-|a|^2)\,I \quad \text{and} \quad \varphi_a'(a) = -(1-|a|^2)^{-1}I,$$

where I is the $n \times n$ identity matrix.

[1] In [11] the function $\sqrt{\rho(x,a)}$ is denoted by $[x,a]$.

Since the following theorem is well known, we state it without proof. A proof may be found in [11, Theorem 3.4.1].

Theorem 2.1.1 *Let ψ be a Möbius transformation of $\hat{\mathbb{R}}^n$ satisfying $\psi(0) = 0$ and $\psi(\mathbb{B}) = \mathbb{B}$. Then $\psi(x) = Ax$ for some orthogonal transformation A.*

We denote by $\mathcal{M}(\mathbb{B})$ the set of all **Möbius transformations of $\mathbb{B}$ onto $\mathbb{B}$**. It is an immediate consequence of the following theorem that the set $\mathcal{M}(\mathbb{B})$ forms a group called the Möbius group of $\mathbb{B}$.

Theorem 2.1.2 *For $a \in \mathbb{B}$, let φ_a be defined by (2.1.6). Then*
(a) *φ_a is a one-to-one Möbius mapping of $\mathbb{B}$ onto $\mathbb{B}$ satisfying*

$$\varphi_a(0) = a, \quad \varphi_a(a) = 0, \quad \textit{and} \quad \varphi_a(\varphi_a(x)) = x$$

for all $x \in \mathbb{B}$.
(b) *If $\psi \in \mathcal{M}(\mathbb{B})$, then there exists an orthogonal transformation A and $a \in \mathbb{B}$ such that $\psi(x) = A\varphi_a(x)$.*

Proof. To prove (a) it only remains to be shown that $\varphi_a(\varphi_a(x)) = x$ for all $x \in \mathbb{B}$. Set $\psi(x) = (\varphi_a \circ \varphi_a)(x)$. Then ψ is a Möbius transformation of $\hat{\mathbb{R}}^n$ mapping $\mathbb{B}$ onto $\mathbb{B}$ satisfying $\psi(0) = 0$. Thus $\psi(x) = Ax$ for some orthogonal transformation A. But then $A = \psi'(0)$. On the other hand, by the chain rule and Equations (2.1.8)

$$\psi'(0) = \varphi_a'(a)\varphi_a'(0) = I.$$

Hence $A = I$ and thus $\varphi_a(\varphi_a(x)) = x$ for all $x \in \mathbb{B}$.
(b) Let $\psi \in \mathcal{M}(\mathbb{B})$ and let $a = \psi^{-1}(0)$. Then $\psi \circ \varphi_a$ is a Möbius transformation of $\mathbb{B}$ that fixes the origin. Thus $\psi \circ \varphi_a(x) = Ax$ for some orthogonal transformation A. But then by (a) we have $\psi(x) = A\varphi_a(x)$. □

Prior to introducing the hyperbolic metric on $\mathbb{B}$ we prove an identity for mappings $\psi \in \mathcal{M}(\mathbb{B})$.

Theorem 2.1.3 *If $\psi \in \mathcal{M}(\mathbb{B})$, then for all $x, y \in \mathbb{B}$,*

$$\frac{|\psi(x) - \psi(y)|^2}{(1 - |\psi(x)|^2)(1 - |\psi(y)|^2)} = \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}.$$

Proof. Although this identity could be proved using the mappings φ_a, it appears to be easier to use the mappings σ_a defined as follows: for $a \in \mathbb{B}$, $a \neq 0$, let σ_a denote the inversion in the sphere $S(a^*, \sqrt{|a^*|^2 - 1})$, that is,

$$\sigma_a(x) = a^* + (|a^*|^2 - 1)(x - a^*)^*. \tag{2.1.10}$$

Then $\sigma_a(0) = a$, $\sigma_a(a) = 0$, and since σ_a is an inversion, $\sigma_a(\sigma_a(x)) = x$ for all $x \in \hat{\mathbb{R}}^n$. Also, by identity (1.3.2),

$$|\sigma_a(x)|^2 = |\sigma_a(x) - \sigma_a(a)|^2 = \frac{(|a^*|^2 - 1)^2 |x - a|^2}{|x - a^*|^2 |a - a^*|^2},$$

which upon simplification gives

$$|\sigma_a(x)|^2 = \frac{|x - a|^2}{\rho(x, a)}.$$

Thus

$$1 - |\sigma_a(x)|^2 = \frac{(1 - |x|^2)(1 - |a|^2)}{\rho(x, a)}. \tag{2.1.11}$$

Hence $\sigma_a \in \mathcal{M}(\mathbb{B})$.[2] Again by (1.3.2) we obtain

$$|\sigma_a(x) - \sigma_a(y)|^2 = \frac{(1 - |a|^2)^2 |x - y|^2}{\rho(x, a)\rho(y, a)}.$$

Combining this with (2.1.11) now gives

$$\frac{|\sigma_a(x) - \sigma_a(y)|^2}{(1 - |\sigma_a(x)|^2)(1 - |\sigma_a(y)|^2)} = \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}.$$

Finally, as in the proof of Theorem 2.1.2(b), every $\psi \in \mathcal{M}(\mathbb{B})$ can be expressed as $\psi(x) = A\sigma_a(x)$ for some $A \in O(n)$ and $a \in \mathbb{B}$. From this the result now follows. □

As a consequence of the identity in Theorem 2.1.3, for $\psi \in \mathcal{M}(\mathbb{B})$,

$$\lim_{y \to x} \frac{|\psi(y) - \psi(x)|}{|y - x|} = \frac{1 - |\psi(x)|^2}{1 - |x|^2}. \tag{2.1.12}$$

This result will be required in proving the $\mathcal{M}$-invariance of the hyperbolic metric on $\mathbb{B}$.

2.2 The Hyperbolic Metric on $\mathbb{B}$

The element of arclength ds for the **hyperbolic metric** d_h on $\mathbb{B}$ is given by

$$ds = \frac{2|dx|}{1 - |x|^2}. \tag{2.2.1}$$

Thus if $\gamma : [0, 1] \mapsto \mathbb{B}$ is a C^1 curve in $\mathbb{B}$, the **hyperbolic length** $L(\gamma)$ of γ is given by

[2] Even though the mappings σ_a are easier to work with, they have the disadvantage that $\lim_{a \to 0} \sigma_a(x)$ does not exist.

$$L(\gamma) = \int_0^1 \frac{2|\gamma'(t)|\,dt}{1 - |\gamma(t)|^2},$$

and for $a, b \in \mathbb{B}$, the **hyperbolic distance** $d_h(a, b)$ between a and b is defined by

$$d_h(a, b) = \inf_{\gamma} L(\gamma),$$

where the infimum is taken over all C^1 curves $\gamma : [0, 1] \mapsto \mathbb{B}$ with $\gamma(0) = a$ and $\gamma(1) = b$. From this we immediately obtain that for $x \in \mathbb{B}$,

$$d_h(0, x) = \log\left(\frac{1 + |x|}{1 - |x|}\right). \tag{2.2.2}$$

Theorem 2.2.1 *For all $\psi \in \mathcal{M}(\mathbb{B})$ and $a, b \in \mathbb{B}$, $d_h(\psi(a), \psi(b)) = d_h(a, b)$.*

Proof. To prove the theorem it suffices to prove that $L(\psi \circ \gamma) = L(\gamma)$ for all C^1 curves γ and $\psi \in \mathcal{M}(\mathbb{B})$. If we set $\sigma(t) = \psi(\gamma(t))$, then σ is a C^1 curve and

$$\begin{aligned}|\sigma'(t)| &= \lim_{h \to 0} \left| \frac{\sigma(t + h) - \sigma(t)}{h} \right| \\ &= \lim_{h \to 0} \frac{|\psi(\gamma(t + h)) - \psi(\gamma(t))|}{|h|},\end{aligned}$$

which by (2.1.12)

$$= |\gamma'(t)| \left(\frac{1 - |\psi(\gamma(t))|^2}{1 - |\gamma(t)|^2}\right).$$

Thus

$$\frac{|\sigma'(t)|}{1 - |\sigma(t)|^2} = \frac{|\gamma'(t)|}{1 - |\gamma(t)|^2}.$$

From this it now follows that $L(\sigma) = L(\gamma)$, thus proving the claim. □

As a consequence of (2.2.2) and Theorem 2.2.1, for $a, b \in \mathbb{B}$,

$$d_h(a, b) = d_h(0, \varphi_a(b)) = \log\left(\frac{1 + |\varphi_a(b)|}{1 - |\varphi_a(b)|}\right). \tag{2.2.3}$$

Some brief computations also give

$$\sinh^2 \tfrac{1}{2} d_h(a, b) = \frac{|a - b|^2}{(1 - |a|^2)(1 - |b|^2)}, \tag{2.2.4}$$