THE ABSOLUTE RELATIONS OF TIME AND SPACE

PRELIMINARY CONSIDERATIONS

THE study of Time and Space is one which in certain respects is extremely elusive and involves a number of difficulties which in ordinary daily life we are apt to overlook.

In scientific work, however, it is all-important to have clear ideas and to know exactly what our statements mean.

This is by no means always an easy task, for it frequently happens that our crude ideas on certain things may be sufficiently precise for certain purposes, but not precise enough for others.

Thus in the ordinary elementary teaching of plane geometry there are certain difficulties which are generally passed over, largely because they are real difficulties and a proper understanding of them could hardly be expected from a beginner.

For instance the use of ruler and compasses and the method of superposition.

The use of the ruler conveys a somewhat crude idea of what we mean in the physical world by points lying in a straight line, while the use of compasses conveys an equally crude idea of what we mean by points in a plane being equally distant from a given point in the plane.

The method of superposition involves ideas which are closely akin to those involved in the supposed use of compasses, but of a more elaborate character.

Both sets of ideas may be described as *ideas of congruence.*

Although there are other difficulties besides these to be overcome, still these will suffice for our present purpose, which is to show that certain points have been slurred over when we first began the study of geometry, which later on may require further elucidation.

Now let us approach this subject as a beginner of sufficient intelligence might be supposed to do.

There is one thing which we might observe, namely: that though we make use of figures drawn on paper to assist us in keeping the facts in mind, yet in proving a theorem, as distinguished from making use of the result, there is no necessity that the figure should be accurately drawn. A very rough figure will suffice and, if we are fairly expert, and the theorem not too complicated, we can dispense with a figure altogether.

Next let us suppose the figures to be accurately drawn on a plane sheet of paper (whatever the expressions "accurately drawn" and "plane" may mean) and then suppose the sheet of paper to be rolled up into a spiral, we could still make use of the figures on the curved sheet as mental images in proving our theorems, although our original straight lines would now (with certain exceptions) be no longer straight.

We could however substitute for our ruler a flexible string, drawn taut, so as to lie in contact with the curved surface of the paper and similarly we could make use of a flexible inextensible tape line or string instead of our original compasses and all our theorems would work out as before, except that lines would be curved which had originally been straight and lengths would be measured along such lines instead of "directly" between points.

With such modifications, to every theorem concerning figures on the plane sheet there will be a corresponding theorem concerning figures on the curved sheet and *vice versa*, and similar methods of proof may be employed in the two cases.

Though the objects about which we are reasoning in the two cases are different, yet the logical processes are formally the same.

We can, however, go still further and consider the case where the figures are accurately drawn (whatever that may mean) on a plane sheet of india-rubber which is then stretched in any way.

In this case straight lines on the unstretched rubber would become lines, straight or curved, on the stretched rubber and a closed curve such as a circle would remain closed after the stretching.

Further, curves which intersected would still intersect and curves which did not intersect would not intersect after the stretching.

A point which lay inside a closed curve such as a circle, would become a point inside a closed curve on the stretched rubber.

Again, a point which lay between two other points in a line of some sort on the unstretched rubber would become a point between two corresponding points on the corresponding line on the stretched rubber.

The distances between the points would of course have altered according to our original standard and two lengths which were originally equal might no longer be equal, but nevertheless certain correspondences would still hold and could be traced between theorems involving equality of lengths on the unstretched rubber and theorems on the stretched rubber.

Perhaps the simplest way of seeing this is to introduce a system of coordinates (say Cartesian coordinates) on the unstretched rubber, by which any point of it would be represented by two numbers.

If then we imagine the rubber to be stretched, the same pairs of numbers could be taken to represent the same points of the rubber after stretching as before. The axes would now, generally speaking, become curved lines and the parallels to them would also in general become curved lines.

The points equidistant from a given point on the unstretched rubber would lie in a circle, and if the equation of this circle be taken as

$$(x - a)^2 + (y - b)^2 = r^2,$$

then this equation would represent also some curve on the stretched rubber. The radii of the circles would become some sort of lines all passing through one point and intersecting the distorted circle.

We should in this way get lines which had been straight, curves which had been circles, lengths which had been equal, etc., and we could deal with these algebraically in the same way as we did with the straight lines, circles and equal lengths on the unstretched rubber.

We notice that the things which actually do remain permanent are the particles of the rubber and certain features of their order.

If we consider the coordinate system we observe that, although the axes and the parallels to them are in general no longer straight after the stretching, yet as either set of parallel lines did not intersect before stretching, so the corresponding lines do not intersect after stretching and they preserve their original order.

We know however that, after a proper foundation has been laid, any geometrical theorem may be proved by coordinate methods and so it is evident that all reasoning which is done after coordinates have once been introduced will apply equally in dealing with certain other things than lines which are truly straight and lengths which are truly equal.

Thus though the sheet of rubber may have originally been plane, yet after stretching it may be curved in innumerable different ways and yet there are certain features which remain invariant throughout.

It is thus evident that although for purposes of mathematical reasoning the actual straightness of lines or actual equality of lengths in the ordinary sense of the terms is not essential, yet when we wish to make use of geometry to describe the physical world the meanings of "straightness" and "equality of length" are all-important.

It is not sufficient that we should say that "there are such things as straight lines," or that "there are lengths which are equal," but it is necessary to have criteria by which we can say (at least approximately) "here are points which lie in a straight line" and "here is a length which is equal to yonder length."

The ruler and compasses give us rough standards of straightness and equality of length in the sense in which these terms are used in ordinary life, but, if we wish to go in for extreme accuracy, other standards must be employed and we must get more precise ideas as to what we really wish to convey when we make use of such expressions.

Consider first the question of what we mean when we say that two bodies are of equal length.

The ordinary method of comparing them is to make use of a measuring rod which we regard as *rigid*; or an *inextensible* tape line. But what do we mean by these words "*rigid*" and "*inextensible*"?

We find that it is by no means easy to say exactly what we do mean.

Approximate rigidity and inextensibility are common enough properties of solid bodies, but by the application of force all bodies are found to be more or less elastic, while change of temperature will also change the length of a rod compared with a parallel rod.

Again, if we wish to compare lengths which are not parallel, the usual mode of procedure would be to turn a measuring rod round from parallelism with the one length into parallelism with the other.

The possibility then arises that during the motion the standard may alter and give us results which indicate the lengths as equal when in reality (whatever that may mean) they are different.

Thus for example, if we wished to compare the lengths OA and OB where A and B are, say, the extremities of the major and minor axes of an ellipse whose centre is O, and suppose we had an elastic tape line which we place first with one end at O and the other at B.

If then keeping the one end fixed at O we move the other round the ellipse we should apparently get the same length for OA as for OB.

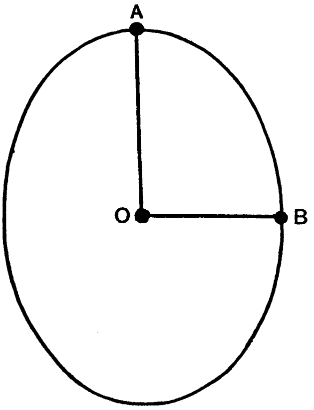

Fig. 1

Now although this seems fantastic, yet the famous experiment of Michelson and Morley seemed to show that just this sort of thing did happen when a body was turned round from a position such that its length was parallel to the direction of the earth's motion in its orbit into a position such that its length was perpendicular to that direction.

The experiment, which was an optical one, consisted in dividing a beam of light into two portions which travelled, the one in one direction, and the other in a transverse direction, and were reflected back again by mirrors.

If we adopt ordinary ideas for the moment and suppose the light to be propagated with a velocity v through a medium and the apparatus to move through that medium with a velocity u, it is easy to calculate the time of the double journey for the two portions of the beam.

For the case of a part of the beam which travels in the direction of motion of the apparatus and back again the time of the double journey is found to be

$$t_1 = \frac{2va_1}{v^2 - u^2},$$

where a_1 is the distance between the point of the apparatus where the beam divides and the corresponding reflector. For the case of the transverse portion of the beam the time of the double journey is found to be

$$t_2 = \frac{2a_2}{\sqrt{v^2 - u^2}},$$

where a_2 is the distance between the point of the apparatus where the beam divides and the other reflector.

Now it is possible to arrange things so that $t_1 = t_2$ and this can be done with extreme accuracy by means of the interference bands which are produced.

We should then have

$$\frac{2va_1}{v^2 - u^2} = \frac{2a_2}{\sqrt{v^2 - u^2}},$$

giving

$$a_1 = \sqrt{1 - \left(\frac{u}{v}\right)^2}\, a_2.$$

Thus a_1 would be slightly less than a_2.

It was found however that, when the apparatus was caused to rotate at a uniform slow rate, and the times of the double journey were equal for one position of the apparatus, then they were equal for all positions. This seemed to indicate that the dimensions of the apparatus in different directions changed as it rotated and the view was put forward by Fitzgerald and Lorentz that a material solid body contracted in the direction of its motion so that a sphere moving through space with a velocity u became a spheroid whose major and minor axes were in the ratio

$$1 : \sqrt{1 - \left(\frac{u}{v}\right)^2},$$

where v is the velocity of light.

It is clear that this once more raises the question as to the real meaning of "equality of length" from which we started out.

Solid bodies apparently do not provide us with standards sufficiently permanent for dealing with such problems.

But the subject of motion raises a number of other difficulties.

There is in particular the question of "absolute motion" and whether this expression has any precise meaning.

The underlying idea of those who believe in "absolute motion" is that, if we consider a definite point of space at any instant, then that point preserves its identity at all other instants. The difficulty of identifying a point of space at two different instants is freely admitted, but for all that (so it is contended) the identity persists.

It was however noticed that, in the classical Newtonian Mechanics, the equations of motion preserved the same form for a system of bodies whose centre of inertia was in uniform motion in a straight line as for a similar system whose centre of inertia was "at rest," so that purely mechanical phenomena could not be expected to show up any difference between the two cases.

The question then naturally arose whether any difference could be detected by optical or electrical means, but experiment failed to show any.

Nextly it was pointed out by Larmor and Lorentz that the electromagnetic equations could also be transformed by a linear substitution so that they preserved the same form for a system moving with uniform velocity as they had for a system "at rest."

In order to do this, however, a "*local time*" had to be introduced.

We are all familiar with the use of "local time" on the earth's surface, but the cases are different in one important respect.

The idea underlying the use of "local time" on the earth's surface is simply that of having different names in different parts of the world for what is regarded as the same instant. Thus noon at Greenwich and noon at New York are both described as 12 o'clock local time, although the instants referred to are clearly different. On the other hand the use of chronometers in navigation is regarded as a method of approximately identifying the same instant at different parts of the earth.

But, as previously remarked, the "local time" used in transforming the electromagnetic equations is of a different character and events which are regarded as simultaneous according to one "local time" would not be simultaneous, in general, when compared by the "local time" of a system which was in motion with respect to the first.

We might of course regard the one "local time" as the true time and the other as a mathematical fiction, but there is no reason known why we should select the one rather than the other, just as there is no way of distinguishing a body "at rest" from one moving uniformly in a straight line.

In fact it appears that, just as we have no method of distinguishing the same point of space at two distinct instants of time, so we cannot strictly identify the same instant of time at two distinct points of space.

It is to be observed that though we started out by trying to give a precise meaning to the idea of equality of length, in which we seemed to be concerned only with space, yet in our attempt to do so, we find difficulties with regard to time intruding themselves.

We can see, however, that even in our original use of compasses the time element intrudes, since in comparing lengths by the use of compasses, the compasses are moved and the idea of motion involves that of time.

Also in the Michelson and Morley experiment, since light takes a finite interval of time in getting from an object P to an object Q and back again to P, we are introducing time relations in comparing lengths.

The question now arises: suppose we imagine a flash of light sent out at an instant A from a particle P to a distant particle Q and arriving there at an instant B and suppose it reflected back to P where it arrives at an instant C; how are we to identify the instant B with any instant at P between A and C?

If we regard P as being "at rest" we might reasonably think to identify B with the instant at P which is midway between A and C, but this would imply that we had some means of measuring intervals of time, and that brings us up against all the same sort of difficulties which we encountered in trying to find a satisfactory method of measuring space intervals.

On the other hand, if P be in uniform motion in the direction PQ it would seem that B is not identical with the instant at P which is midway between A and C.

In any case we do not know of any means of telling whether P is "at rest" or not.

Having thus been led on from the consideration of spacial relations to those of time we seem at first sight to have increased our difficulties instead of solving them, but if we persevere in our task we shall find that we have made an appreciable advance towards solving our problem.

From the consideration of figures drawn on a sheet of rubber which was afterwards stretched in any way, we were led to recognise the importance of *order* in the study of the logic of geometry, and since order also plays a part in time relations, it seems worth while to consider order in time.

Now here we find an interesting and very important thing.

If I consider two distinct instants of which I am *directly conscious**, I notice that the one is *after* the other.

Noon to-day is *after* noon yesterday and I cannot invert the order.

There is in fact what is called an asymmetrical relation between the two instants, such that if B be *after* A, then A is not *after* B.

* The fact that I am *directly conscious* of the two instants is very important, in view of later developments.

If we consider two points or two particles in space, say *P* and *Q*, there is nothing analogous to this and we have no reason to say that *Q* is *after P* rather than that *P* is *after Q*.

We might, of course, give them an order by means of some convention, but such convention would be quite arbitrary, whereas in the case of the instants, it is a matter of fact and not of convention, quite independently of what words we may employ to express that fact.

The simplest relation of order among points is a relation of *between* which involves three terms instead of two.

This relation of *between* has been employed by various mathematicians in investigating the foundations of geometry, but the relation of equality of lengths then appears as something extraneous, grafted on to the system.

The use of an asymmetrical relation such as *after* appears to have great advantages over a relation such as *between* in constructing a theory of order and I have found it possible, by means of such a relation, to construct a system of geometry of space and time. It might perhaps more correctly be described as a geometry of time, of which spacial geometry forms a part.

In constructing this system it is necessary to modify certain currently accepted notions, but the modifications required all appear to be capable of justification and the structure, when completed, will be found closely to resemble our ordinary conceptions.

We shall regard an instant as a fundamental concept which, for present purposes, it is unnecessary further to analyse, and shall consider the relations of order among the instants of which I am directly conscious.

Thus for such instants we find the following properties:

(1) If an instant *B* be *after* an instant *A*, then the instant *A* is not *after* the instant *B*, and is said to be *before* it.

(2) If *A* be any instant, I can conceive of an instant which is *after A* and also of one which is *before A*.

(3) If an instant *B* be *after* an instant *A*, I can conceive of an instant which is both *after A* and *before B*.

(4) If an instant *B* be *after* an instant *A* and an instant *C* be *after* the instant *B*, the instant *C* is *after* the instant *A*.

(5) If an instant *A* be neither *before* nor *after* an instant *B*, the instants *A* and *B* are identical.

The set of instants of which I am directly conscious have thus got a linear order.

But now let us consider the fifth of these properties.

It might at first sight be supposed that it was a necessary consequence of our ideas of *before* and *after*. That it is really logically independent of the other properties may be shown by the help of a geometrical illustration. This illustration is very suggestive and we purpose to make further use of it, but the logic of our theory is independent of the illustration.

Suppose we have a set of cones having their axes parallel and having equal vertical angles, and further, suppose each cone to terminate at the vertex, which is however to be regarded as a point of the cone.

We shall call such a cone having its opening pointed upwards an α cone, and one with the opening pointed downwards a β cone.

Thus corresponding to any point of space there is an α cone of the set having the point as vertex, and similarly there is a β cone of the set having the point as vertex.

Now it is possible by using such cones and making a convention with respect to the use of the words *before* and *after* to set up a type of order of the points of space.

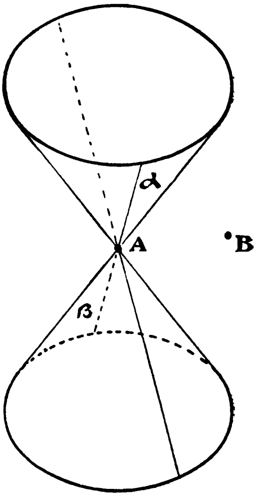

Fig. 2

For the purposes of this illustration we shall make the convention that, if A_1 be any point and α_1 and β_1 be the corresponding α and β cones, then any point A_2 will be said to be *after* A_1 provided it is distinct from A_1 and lies either on or inside the cone α_1 and will be said to be *before* A_1 provided it is distinct from A_1 and lies either on or inside the cone β_1.

It is easy to see that with this convention we have the following:

(1) If a point B be *after* a point A, then the point A is not *after* the point B and is said to be *before* it.

(2) If A be any point, there is a point which is *after* A and also a point which is *before* A.

(3) If a point B be *after* a point A there is a point which is both *after* A and *before* B.