I

PLANIMETRY ON A MOVING PLANE.

[*Nature*, Oct. 27, 1881.]

PROF. MINCHIN'S theorem in *Nature* [which is also M. Darboux's, namely, "If a plane, *A*, move about in any manner over a fixed plane, *B*, and return to its original position after any number of revolutions, all those right lines in the plane *A* which have enveloped glisettes of the same area, are tangents to a conic, and by varying the area of the glisette we obtain a series of confocal conics"] may be proved easily by considering the motion as due to the rolling of one closed curve on another back into its first position, their lengths being of course commensurable. If you measure y for the rolling curve from the straight line which forms the envelope, and x along that line, then the differential of the area between the envelope and the fixed curve is easily seen to be $ydx + \frac{1}{2}y^2d\omega$, where $d\omega$ is the angle turned through by the rolling curve, and is equal to ds multiplied by the sum of the curvatures at the point of contact, which we shall call σ. The summation of the former part is a multiple of the area of the rolling curve, and therefore the same for all lines; that of the latter is half the moment of inertia of matter distributed over its perimeter with density σ, about the line in question. The result is therefore the well-known property of equi-momental ellipses. Similar reasoning, with the use of the property of the centre of inertia of a system, leads to the further result that when the perimeter of the envelope is of constant length, the line touches a circle, and different values of the constant correspond to concentric circles. In the same way by a property of the centre of inertia we may also prove immediately the known theorem that when the area traced out by a point is constant, the point lies on a circle, and different values of the constant correspond to concentric circles; and we may extend it to areas traced on a sphere.

Lines enveloping equal areas:

equal perimeters.

Points tracing equal areas.

2

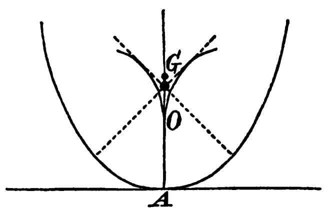
ON CRITICAL OR "APPARENTLY NEUTRAL" EQUILIBRIUM.

[*Proc. Camb. Phil. Soc.* Vol. IV. Pt VI. (May 28, 1883).]

1. When a solid body is resting on a fixed surface, its equilibrium is stable when its centre of gravity is vertically below the centre of curvature at the point by which it rests, and unstable when vertically above it: when the two points coincide the equilibrium is often said to be apparently neutral, and its real character is discriminated by an analysis of the differentials of higher orders. It may be worth while to trace the origin of this peculiarity, and its practical effect on the nature of the equilibrium in cases which approximate to this critical condition.

Range of instability:

2. Let us take the case of a heavy body symmetrical about two principal planes through its axis AB (one of them the plane of the figure), and resting on a horizontal plane at A. The evolute of the section has a cusp at O, the centre of curvature corresponding to A. Let us suppose it to point downwards, so that the radius of curvature is a minimum at A, and let us suppose the centre of gravity G to be a very short distance above O. The position of the body is unstable, but a stable position exists in immediate proximity on each side, in which the tangents from G to the evolute are vertical. We see therefore, that when left free the body will oscillate at first round its upright position, and will finally settle down in one of these two slightly inclined positions. When G moves down to O, these two flanking stable positions come nearer to the upright position, and finally come up to it, so that the equilibrium is really stable. But there is this peculiarity, that its oscillations round the vertical are no longer approximately Simple Harmonic, but follow another law which we can easily investigate, and that they are executed with extreme slowness: and we can trace the change to this new law from the rocking motion which is compounded of oscillations round the two flanking stable positions alternately.

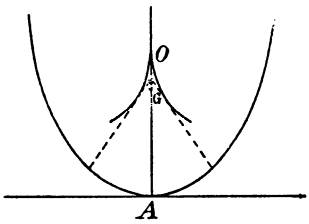


how it closes up.

Range of stability:

3. If the cusp pointed upwards, and G were a very short distance below O, we would have a vertical stable position flanked by two

very near positions of instability: and so, when G moves up to O, the vertical position becomes unstable. It is important, then, to bear in mind, that cases which satisfy the condition of stability, but are near to the critical case, are practically unstable for oscillations of any considerable amount, when the radius of curvature is a maximum at A.



how it closes up.

4. These considerations clearly apply to all cases of critical or "apparently neutral" equilibrium, so that the determination of its real character carries with it the determination of the *practical* character of all other cases which approximate to that condition.

5. In the case of a floating body this discrimination is easy. If we consider, as usual, oscillations in which the displacement is constant, the centre of gravity of the displacement traces out a surface called the Surface of Buoyancy, and we know that the tangent plane to this surface corresponding to any position of the oscillating body is always horizontal, and that therefore the resultant fluid pressure acts along the normal at its point of contact. The circumstances of the oscillation are therefore the same as those of a solid body with the same centre of gravity, but bounded by the surface of buoyancy, and rolling on a frictionless horizontal plane. The evolute of the section of the surface of buoyancy is the locus of the metacentre, and has been called the Curve of Stability: and a case of equilibrium which approximates to the critical condition will come under § 2 or § 3 according as this curve has its cusp pointing downwards or upwards. Now the radius of curvature of the surface of buoyancy is known by the ordinary theory to be equal to the moment of inertia of the corresponding plane of floatation about an axis through its centre of gravity perpendicular to the plane of displacement, divided by the volume of the displacement: and therefore the case comes under § 2 or § 3 according as this moment of inertia increases or diminishes as the degree of heeling increases, a criterion usually easy of application.

Range of stability of a ship: the criterion:

6. In a very numerous class of cases we can completely determine by geometry the curves mentioned above. Since any surface of the second degree may be derived by successive orthogonal projections, real or imaginary, from a sphere, it follows that the locus of the centre of gravity of a constant volume cut off from it by a plane is a similar and concentric surface. Hence for all this class of surfaces, which includes quadrics, cones, cylinders, pairs of planes, the surface of

buoyancy is similar to the bounding surface of the floating body. Now the radius of curvature of a parabola or hyperbola is least at the vertex, and that of an ellipse is least at the ends of the major axis, and greatest at the ends of the minor axis: hence for paraboloidal and hyperboloidal surfaces, and cones, and wedges, the critical position is really stable, and for surfaces of which the section in the plane of displacement is elliptic it is stable or unstable according as the major or minor axis is vertical.

results for a class of cases.

Further, if the shape near the water line come under any of these heads, the above conclusions clearly all apply, irrespective of the shapes of the other parts of the body.

Case of parabolic section.

7. We shall now investigate, as a typical case, the nature of the oscillations of a body whose section at the vertex approximates most closely to that of the parabola $y^2 = ax$, which rolls on a rough plane, and in which G is a very short distance c above O.

The equation of the evolute near the cusp is of the form

$$y^2 = \frac{16}{27a} x^3,$$

and when it is displaced through a small angle θ, the distance of the vertical tangent to the evolute from G is easily seen to be equal to

$$\frac{a}{4} \theta^3 - c \sin \theta,$$

to the third order in θ.

Thus, if κ represent the radius of gyration of the solid round an axis through its vertex, the equation of motion is

$$\frac{\kappa^2}{g} \frac{d^2\theta}{dt^2} = c\left(\theta - \frac{\theta^3}{6}\right) - \frac{a}{4} \theta^3,$$

to the third order in θ.

And the flanking positions of stable equilibrium are given by

$$\theta = \sqrt{12c/(3a + 2c)}.$$

This equation is of the form

$$\frac{\kappa^2}{g} \frac{d^2\theta}{dt^2} = c\theta - m\theta^3,$$

where c is very small compared with m, which may be put equal to $\frac{1}{4}a$; and in fact we are investigating the character of oscillations under a restoring force proportional to the cube of the distance disturbed by a small force proportional to the distance.

8. We have

Oscillations near the critical stage.

$$\frac{\kappa^2}{g} \left(\frac{d\theta}{dt}\right)^2 = \frac{1}{2} m\beta^4 - c\beta^2 + c\theta^2 - \frac{1}{2} m\theta^4,$$

where β is the amplitude of the swing; and therefore

$$t = \int \frac{\kappa d\theta}{g^{\frac{1}{2}}\sqrt{A + c\theta^2 - \frac{1}{2}m\theta^4}}$$

$$= \frac{\kappa}{\sqrt{Ag}} \int \frac{d\theta}{\sqrt{(1 - p\theta^2)(1 + q\theta^2)}}$$

$$= \frac{\kappa}{\sqrt{Apg}} \sqrt{1 - k^2} \int \frac{-d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},$$

where $\cos\phi = \sqrt{p}\,\theta, \quad k^2 = \frac{q}{p+q},$

$$A = \tfrac{1}{2}m\beta^4 - c\beta^2,$$

$$q - p = \frac{c}{A}, \text{ which is small,}$$

$$qp = \frac{m}{2A};$$

so that $q + p = \sqrt{\frac{2m}{A}}\left(1 + \frac{c^2}{4mA}\right)$, approximately.

Also $\frac{c}{m\beta^2}$ is a small quantity which we shall call e.

Hence $t = \frac{\kappa}{g^{\frac{1}{2}}\sqrt{A(p+q)}} \int \frac{-d\phi}{\sqrt{1 - k^2 \sin^2\phi}}$

$$= \kappa\,(mg\beta^2)^{-\frac{1}{2}} \left(1 + \frac{c}{2m\beta^2}\right) \int \frac{-d\phi}{\sqrt{1 - k^2\sin^2\phi}}$$

$$= \kappa\,(mg\beta^2)^{-\frac{1}{2}} (1 + \tfrac{1}{2}e) \left\{F\left(\frac{\pi}{2}, k\right) - F(\phi, k)\right\};$$

where $\cos\phi = \sqrt{p}\,\theta = \frac{\theta}{\beta}(1 - \tfrac{1}{2}e), \quad k^2 = \tfrac{1}{2}(1 + e).$

Now write $\cos\psi = \frac{\theta}{\beta}$, and use the results for differentiating F that are given in Cayley's *Elliptic Functions*, § 73. We find, on putting $\phi = 0$, that a quarter period of the oscillation of amplitude β is given by

$$T = \kappa\,(mg\beta^2)^{-\frac{1}{2}} (1 + \tfrac{1}{2}e) \left[F\left(\frac{\pi}{2}, \frac{1}{\sqrt{2}}\right) + e\left\{E\left(\frac{\pi}{2}, \frac{1}{\sqrt{2}}\right) - \tfrac{1}{2}F\left(\frac{\pi}{2}, \frac{1}{\sqrt{2}}\right)\right\}\right]$$

$$= \frac{\kappa}{g^{\frac{1}{2}} m^{\frac{1}{2}} \beta}[F_1 + eE_1],$$

in which $F_1 E_1$ stand for the complete elliptic integrals of the first and second orders to modulus sin 45°, and $e = c/m\beta^2$.

Also

$$t = T - \frac{\kappa}{g^{\frac{1}{2}} m^{\frac{1}{2}} \beta} (1 + \tfrac{1}{2}e) \left[F + e(E - \tfrac{1}{2}F) - \tfrac{1}{2} e \sin\psi \cos\psi + \frac{e \cos\psi}{2 \sin\psi \sqrt{1 - \frac{1}{2}\sin^2\psi}} \right]$$

$$= T - \frac{\kappa}{g^{\frac{1}{2}} m^{\frac{1}{2}} \beta} \left[F + eE - \frac{e}{2\beta^2} \theta \sqrt{\beta^2 - \theta^2} + \frac{e\beta\theta}{\sqrt{2(\beta^4 - \theta^4)}} \right],$$

in which F, E are the functions of $\text{arc} \cos \dfrac{\theta}{\beta}$ to modulus $\sin 45°$, whose values can be taken at once from Legendre's tables.

9. The disturbance produced by the small term proportional to the distance is represented by the terms multiplied by e. And, in particular, if $e = 0$, so that the equilibrium is critical, we have

The critical oscillations.

$$T = \frac{\kappa . F\left(\frac{\pi}{2}, \sin 45°\right)}{g^{\frac{1}{2}} m^{\frac{1}{2}} \beta},$$

$$t = \frac{\kappa}{g^{\frac{1}{2}} m^{\frac{1}{2}} \beta} \left\{ F\left(\frac{\pi}{2}, \sin 45°\right) - F\left(\text{arc} \cos \frac{\theta}{\beta}, \sin 45°\right) \right\},$$

where $m = a/4$, $\beta =$ amplitude of excursion.

In this case, therefore, the period of an oscillation varies inversely as its amplitude, and is equal to

$$\frac{2\kappa}{g^{\frac{1}{2}} a^{\frac{1}{2}} \beta} \times 1{\cdot}85407.$$

If the plane on which the motion takes place be frictionless, κ will be the radius of gyration about an axis through the centre of gravity: and the same will be the case in the problem of hydrostatic oscillations.

10. In the case of one sphere resting on another it is very easy to form the equation of energy, and thence determine the value of m.

Body rolling on another.

In the case of one symmetrical body resting on another, whether on the summit or not, we have, if θ be the angle by which G overhangs the vertical through the point of support,

$$\frac{\kappa^2}{g} \frac{d^2\phi}{dt^2} = \left(\frac{d^3\theta}{ds^3}\right)_0 \frac{s^3}{6}$$

in the critical case; therefore

$$\frac{d^2 s}{dt^2} = \frac{g}{6\kappa^2} \left(\frac{1}{\rho} + \frac{1}{\rho'}\right)^{-1} \left(\frac{d^3\theta}{ds^3}\right)_0 s^3,$$

in which the value of $\left(\frac{d^3\theta}{ds^3}\right)_0$ is given by Mr Routh in the *Quarterly Journal of Mathematics*, XI, p. 106; and thus the value of m is determined. The nature of the equilibrium in the critical case appears to have been first discussed by Dr A. H. Curtis, *Q. J.* IX, p. 42.

11. The exception to this critical case again, is that in which five positions of equilibrium come together. Then the equation of motion is of the form **Higher critical circumstances.**

$$\frac{d^2\theta}{dt^2} = -\mu\theta^5.$$

Hence
$$t = \sqrt{\frac{3}{\mu}}\int\frac{d\theta}{\sqrt{\beta^6 - \theta^6}}$$
$$= \sqrt{\frac{3}{\mu}}\frac{1}{\beta^2}\int\frac{d\phi}{\sqrt{1 - \phi^6}}.$$

And on writing
$$\phi^2 = \frac{1}{1 + y^2},$$
$$y = 3^{\frac{1}{4}}\tan\frac{\psi}{2},$$

this reduces to (*vide* Bertrand's *Integral Calculus*)

$$t = \sqrt{\frac{3}{\mu}}\frac{1}{\beta^2}\frac{1}{3^{\frac{1}{4}}}\int\frac{d\phi}{\sqrt{1 - \sin^2 15^\circ \sin^2\psi}},$$

and a quarter period is given by

$$T = 2\frac{3^{\frac{1}{4}}}{\mu^{\frac{1}{2}}\beta^2} \times 1{\cdot}59814,$$

which varies inversely as the square of the amplitude.

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3

ELECTROMAGNETIC INDUCTION IN CONDUCTING SHEETS AND SOLID BODIES.

[*Philosophical Magazine*, Jan. 1884.]

Electro-dynamic flow in unlimited sheets: I. The problem of electromagnetic induction in continuous conductors has often engaged attention. Maxwell in 1871 (*Proc. Roy. Soc.*, and "Electricity and Magnetism") gave the complete solution for the case of an infinite sheet, by means of a moving trail of images of the inducing disturbance. He refers to Jochmann (Crelle's *Journal*, 1866), who had investigated the steady currents that would be set up in an infinite plane or a spherical conductor, rotating uniformly in a field of force, on the supposition that the effect of this external field on the conductor is very great compared with that of the system of currents excited. This, we shall find, presupposes a velocity of all the parts of the conductor in the immediate neighbourhood of the seat of the disturbance which must not exceed a certain moderate limit, or else a correspondingly great resistance.

Maxwell's image-system:

Jochmann's approximation for spherical case:

C. Niven's analytic solutions. In the *Philosophical Transactions* for 1880 Professor Charles Niven has developed a complete mathematical solution of Maxwell's equations for infinite plane sheets and spherical sheets of any thickness; but he has not fully discussed any of the simpler cases. The generality of the solution necessarily complicates his expressions; and in what follows we shall, to secure compactness and completeness, work out *ab initio* the results that we require. The methods are chosen with a view to deducing the results directly from consideration of the physical quantities involved, with as little appeal to mathematical transformations as possible.

The solutions given below are all for cases that can be presented in a simple form. A principal object in obtaining them is to examine the circumstances of what may be called electromagnetic screens.

Electro-magnetic screening: It has long been known that a plate of soft iron placed in front of a magnet will partially screen the space on the other side of the plate from the influence of the magnet, and that the screening increases with the thickness of the plate. Sir William Thomson has applied the effect in recent times by enclosing his marine galvanometer in a heavy soft-iron case to protect it from the magnetism of the iron of the ship. And lately Stefan has published a paper in

Wiedemann's *Annalen*, in which he investigates experimentally the diminution of the strength of a magnetic field which is produced in the interior of soft-iron cylinders, and compares the results with the indications of Poisson's theory of induced magnetization.

Now if we have a sheet of conducting matter in the neighbourhood of a magnetic system, the effect of a disturbance of that system will be to induce currents in the sheet of such kind as will tend to prevent any change in the conformation of the tubes of force cutting through the sheet. This follows from Lenz's law, which itself has been shown by Helmholtz and Thomson to be a direct consequence of the conservation of energy. But if the arrangement of the tubes *in* the conductor is unaltered, the field on the other side of the conductor into which they pass (supposed isolated from the outside spaces by the conductor) will be unaltered. Hence if the disturbance is of an alternating character, with a period small enough to make it go through a cycle of changes before the currents decay sensibly, we shall have the conductor acting as a screen.

from alternating field by conducting sheet:

Further, we shall also find, on the same principle, that a rapidly rotating conducting-sheet screens the space inside it from all magnetic action which is not symmetrical round the axis of rotation.

from steady field by spinning sheet:

The earth considered as a rotating body comes under this case; for the upper strata of the atmosphere are conductors of electricity, whether the conduction follows the law of Ohm or not, and therefore these principles show that this layer must more or less protect the interior from any external magnetic action, not symmetrical round the Earth's axis, that might exist outside it. In the case of a spherical sheet conducting according to Ohm's law, we shall find that the screening action depends upon the angular velocity multiplied by radius divided by the resistance of the sheet, so that defect of conductivity is fully made up for by a large radius.

e.g. atmospheric sheet around the Earth, is effective.

We shall also find expressions for the magnetic moment of a copper sphere or spherical shell rotating in a magnetic field, and for the expenditure of power required to keep up the motion, and also for the rate of damping of the oscillations of such a body.

Spinning globes and discs in magnetic field.

Finally, we shall show how solutions may be obtained for the case of a rotating circular disc of conductivity not very large by neglecting the mutual action of the induced currents.

II. In the case of a conductor rotating steadily in a magnetic field, which is that with which Jochmann deals, a steady distribution of currents in space will ensue when the conductor is symmetrical about the axis of rotation; and the electromotive force along any line will be given, according to Faraday's rule, by the number of

No currents induced by a symmetrical field:

tubes of force of this steady field that are cut through by the line per second. Now when the magnetic field is symmetrical round the axis of rotation, the number of tubes enclosed in any closed moving circuit in the conductor will not alter at all, so that there will be no current round any circuit, and therefore no induced current whatever: the electromotive force along each open line will accumulate a statical electric charge at one end of it, so that the conductor will become electrified until the induced electromotive force is exactly neutralized by the statical difference of potentials. This conclusion holds whatever be the shape of the body.

the annulling slight electrification.

Taking cylindrical coordinates, if α, β, γ be the components of the magnetic force at the point $r\theta z$ in the directions of dr, $rd\theta$, dz respectively, the electromotive forces between the ends of these elements of length will be $-\omega r\gamma dr$, 0, $\omega r\alpha dz$, where ω is the angular velocity, by Faraday's rule; and thus the electrostatic potential which neutralizes these will be

$$\psi = \int(\omega r\gamma dr - \omega r\alpha dz)$$

$$= \omega \int r\,(\gamma dr - \alpha dz).$$

From this the electrification of the conductor may be deduced at once. For example, taking a uniform field $\gamma = \gamma_0$ parallel to the axis of rotation, we have

$$\psi = C + \tfrac{1}{2}\omega\gamma_0 r^2,$$

and therefore there is a uniform volume density of electricity

$$-\frac{1}{4\pi}\nabla^2\psi = -\frac{\omega\gamma_0}{2\pi}.$$

We must add the proper surface distribution: for instance, if the conductor be a sphere, the outside potential which corresponds to the given value at the surface is

$$\psi_1 = C\frac{a}{\rho} - \tfrac{1}{2}\omega\gamma_0 a^5\,\frac{3\cos^2\phi - 1}{3\rho^3} + \tfrac{1}{3}\omega\gamma_0\frac{a^3}{\rho},$$

where ρ is the radial line, and ϕ its inclination to the axis. Thus the surface density

$$= \frac{1}{4\pi}\left(\frac{d\psi}{d\rho} - \frac{d\psi_1}{d\rho}\right)$$

$$= \frac{\omega\gamma_0 a}{8\pi}\left(-\tfrac{4}{3} + 5\sin^2\phi\right).$$

The arbitrary constant C allows us to superpose any free distribution. If the charge of the body was originally zero, we may give it such a value that the charge shall remain zero; but if the axis of rotation is uninsulated, the condition is that C is zero. The above agrees with Jochmann's results.