CHAPTER 1

A Girdle round about the Earth

PEOPLE SOMETIMES imagine mathematicians sitting in armchairs, with heavy tomes about them, solemnly working out abstruse problems of no interest to anyone but themselves. Calm, dispassionate, there they sit, thinking only of numbers and the world of numbers. Dispassionate, for who could be passionate about mere numbers?

Not a bit of it.

People who try to find humdrum explanations of things that are not humdrum, have tried to explain the joy of mathematics by saying that it consists of the mental satisfaction of getting sums right. It makes me laugh, as Herodotus exclaimed two thousand years ago. He knows little of mathematics who thinks that; because mathematics is of all subjects the most adventurous. It adventures alike into the infinitely small and the infinitely great; it is the one subject that can stretch out confidently to the bounds of the universe.

And at any moment new and unsuspected mathematical horizons may open up. Even to follow in the footsteps of the great discoverers is exciting enough. To follow de Moivre through the steps of his exciting theorem, to be suddenly confronted with the whole new world of imaginary trigonometry. To sit with Newton and watch the fall of the famous apple that fired his imagination with the idea that falling apple and falling moon obey a single universe-wide law. With Cavendish to set out to weigh the world in a balance.

In the year 1500 a coach rolled out of Paris on a north and south road. M. Fernel leaned out of the coach with his eyes fixed on one of the carriage wheels. People stared at him; small boys shouted ribald remarks; but his eyes remained serenely glued to the wheel. And his lips moved as he counted, and went on counting. What was this extraordinary man doing? Strange as it may sound, he was measuring the earth. And lest anyone else should think he was as ridiculous as the ribald youths seemed to think, I hasten to add that the result of his measurement was quite good.

After all, since men began to think, who had not wondered about the size of the earth, even if only to dismiss the idea of measuring it as an impossibility? 'Who hath stretched a line upon the earth?' was demanded of Job. There was no answer. It must have seemed an impossible task, so long as the idea of a fixed earth persisted.

It is the rotation of the earth that makes its measurement a comparatively simple operation, and that justified M. Fernel in counting the turns of his carriage wheel. Rotation provides us with a natural framework on which we can make the necessary measurements. It provides us with two fixed points; these are the North Pole and the South Pole, the ends of the axis on which the earth rotates. It provides us also with a fixed circle; this is the equator, the great circle halfway between the poles.

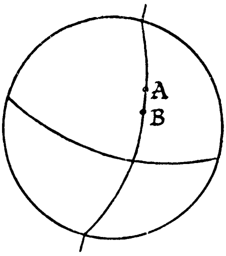
We use the equator as a zero line, and we measure latitudes north and south from it. Measurements of latitude are made in degrees, for the simple and satisfactory reason that it is much easier to measure angles than to measure lengths. The mere turn of a telescope on a graduated circle is usually sufficient to measure an angle; whereas the measurement of a considerable length is a difficult process. Very often indeed a direct measurement of length is impossible. Navigating officers at sea may guess their positions by dead reckoning, based on vague measurements of distance; but after a period of clouded skies they must long for a sight of the punctual stars with their promise of exact angular measurement.

In measuring a length on land we have to make allowances for slopes and inequalities, so that it is advisable to choose the flattest piece of land available. No one, I should imagine, would choose the Himalayas for such a measurement, though on one famous occasion the incurably romantic spirit of French mathematicians drove them to mountainous Peru. Commonsense Englishmen chose a level tract on Salisbury Plain to make an exact measurement of length; Russian mathematicians are fortunate in having vast level plains for their measurements; a north and south line through Paris is level enough; and there are few countries without level stretches to simplify their measurements of length. Indeed, the modern method of making all measurements from a single carefully measured base originated on the frozen meadows of Holland.

In measuring a straight line we have to make sure that the measuring rods are evenly spaced and in an exactly straight line; we have to make allowances for the imperfections of the measuring rods, and especially for their expansion by heating; we have to make allowance for the curvature of the earth on which we are measuring, so that the line may deviate from a strictly straight direction to follow the curvature. The difficulty of measuring a long line following the slight curve of the earth, with anything approaching exactitude, is so great that it is usual to measure one line a mile or more in length, and then to make all the other measurements in angles. The original base line, measured with extreme care, enables all the other lengths to be calculated, by the methods of trigonometry, from the measured angles.

Before we begin to attempt to measure the earth we have to have some idea of its shape. We are measuring, as it were, blindfold; we can see nothing of the actual shape of the world, and very little even of the lines we are measuring. I find it difficult to imagine how a Flat Earthist would set about the job of measuring the earth. Random measurement would take us nowhere in particular; so that even a Flat Earthist must make some kind of assumption about his flat earth before he can begin to measure it.

It is not unreasonable to suppose that the earth is spherical in shape; there have always been the round sun and the equally round moon to suggest such a shape. If we decide that the earth is spherical, we have got something to go on. We might, for example, find two places, *A* and *B*, one of which is exactly north of the other. We measure the length *AB* by the best means at our disposal; we may, with great labour, measure the distance with measuring rods, or we may pace it, or we may rely on the estimates of travellers. Then we need to find the latitudes of *A* and *B*, that is, their angular distances from the equator. Suppose



the difference in latitude is 4°; the distance *AB*, which we have measured, represents 4°. All we have to do is to find what distance represents 360°; that is, we multiply the distance *AB* by $\frac{360}{4} = 90$.

The ancient Greeks had clear ideas about the shape of the earth; they imagined it to be spherical, and so big that the heights of mountains made no appreciable difference to the rotundity. Further, they were able to measure latitudes by observations of the sun and the stars. Eratosthenes, who flourished, as one says, about 200 B.C., made a measurement of the circumference of the earth; he gave it as 252,000 stadia. A stadium was the length of the track for foot-races; we are more familiar with the word as the name for the terraced seats for spectators round the track—this was a later use of the word. Measurement of the stadium at Athens has shown that the length was 607 feet, so that Eratosthenes' calculation was:

$$252{,}000 \times 607 \text{ feet} = \frac{252{,}000 \times 607}{5280} \text{ miles}$$
$$= \text{about } 29{,}000 \text{ miles.}$$

When we consider the difficulty of measuring latitudes and long distances without the use of precision instruments, that is an extremely good approximation to the circumference of the earth. Eratosthenes got his difference of latitude in a rather extraordinary way. At Syene in Upper Egypt there was a deep well. On one day of the year, the summer solstice, when the sun reaches its highest point in the sky, there was no shadow at the bottom of the well at noon. The sun must then have been exactly overhead. At noon on the same day the sun at Alexandria was 7° 12 from the zenith; so the difference in latitude was 7° 12′. The north and south distance was 5000 stadia, so Eratosthenes had the simple sum:

$$5000 \times \frac{360^\circ}{7^\circ\ 12'} = 5000 \times 50$$
$$= 250{,}000 \text{ stadia.}$$

Later on he added another 2000 stadia to his calculated result and made it that much less accurate.

Eratosthenes' method was correct, and so also was his arithmetic. But he had to rely on the estimates of surveyors for the distance between Syene and Alexandria, and they made the distance about 25 per cent greater than we now know it to be.

Posidonius came later with an estimate of 240,000 stadia, which accidentally came nearer the truth. He too revised his estimate. Unfortunately he reduced it to 180,000 stadia, which is just over 20,000 miles. Unfortunately also the great Ptolemy accepted this shrunken estimate; and the world remained shrunken in men's minds, at any rate in the minds of those who thought about such things, until Columbus startled them into a realisation of the neglected 5000 miles of circumference.

After the Greeks—eclipse. It was not till long afterwards that the Arabs took up the tale. About the year A.D. 800 the Caliph Almamoun decided that the earth should be measured. He carried out his project in a spectacular way worthy of a great caliph. He assembled two parties of astronomers back to back on the plains of Mesopotamia where there were no mountains to interfere with the measurements. The two parties measured the latitude of the starting-point; and then they set off in opposite directions, measuring the distance with rods as they went. One party went north and the other south, and they continued until each had reached a point one degree from the zero point. The double measurement gave them the length of two degrees of latitude, and they had then only to multiply this length by 180 to find the circumference of the world. But it was not an easy task. It is far from easy to measure a distance of about seventy miles with rods, and to keep the measurements in anything like a straight line; it is far from easy to measure exact latitudes with rude instruments. The wily caliph ensured that there should be no 'distribution of errors' or other cooking of results, by segregating the two parties. There was a distance of 140 miles between them when they had done measuring, and the lack of collaboration produced poor results. He was probably a saddened and disillusioned caliph with a much shaken faith in astronomy and mathematics.

Another long eclipse. And then in 1500 Fernel rolled out of Paris in his coach, counting the number of turns of one of the wheels as he went. He had only to measure the circumference of the wheel and multiply it by the number of turns to get a very fair estimate of the distance. As the road ran north and south, he had measured part of a meridian. By a bit of luck he got a good angular measurement too; so he did get a fair result.

In 1581 an infant prodigy was born at Leyden in Holland; he was Willebrod Snell. By the age of twelve he seems to have mastered the mathematics of his day; and he sought other fields. There were plenty around him. The frozen meadows of Holland gave him level ground for a good base line; and he initiated there the modern method of triangulation. He measured a base line, and then found the angular distances of points around it. The rest is a matter of trigonometrical calculation. The calculations are not easy, because allowances have to be made for the curvature of the earth, for ups and downs, and for small inaccuracies of measurement. But they can be made with great accuracy. Snell's innovation made it possible to measure considerable arcs of meridians, and of parallels of latitude, with greater accuracy than was possible, and less labour than was necessary, before his time.

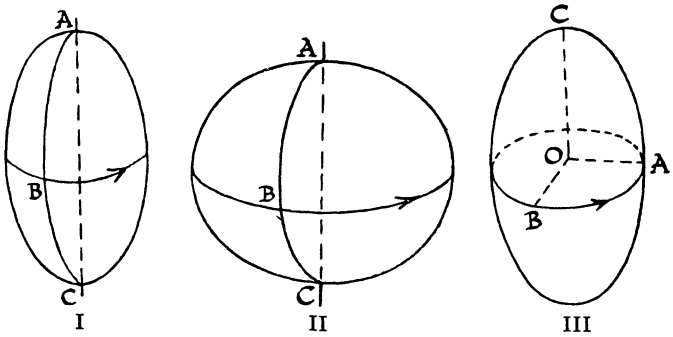
That is just about as far as it was possible to get in earth measurement; or so it seemed. There might be a little increase in exactness, and that is all: from a vague measurement to the nearest thousand miles, then to the nearest hundred, and so on with progressive increases in exactness. So it seemed. Until a French astronomer named Richer went to the island of Cayenne in Guiana. He took with him a clock which accurately beat seconds in Paris. But in Cayenne it went $2\frac{1}{2}$ minutes slow every day. Most travellers would have shortened the pendulum and thought no more about it. Not so an astronomer. Especially when other astronomers, adventuring toward the equator, found the same thing.

There had to be a reason, and it was Newton who saw the reason. The clock went slow because the pendulum fell more slowly than when it was in Paris. That is, the acceleration due to gravity was less, and this could only mean that gravity itself was less. The pendulum actually weighed less near the equator than when it was farther north. That could be due to one of two things, or to both in combination. We might be farther from the centre of the earth at the equator, in which case gravity would be less; or the greater rate of rotation at the equator would increase outward 'centrifugal force' and so ease the downward pressure that is weight. Newton deduced, from the facts, that the earth had an equatorial bulge, and a consequent flattening at the poles.

The earth had ceased to be a sphere. That is of course the conception of the earth in the minds of mathematicians, the conception on which measurements of the earth are made. A fresh model of the earth had to be found.

If we cut out such ridiculous shapes as approximate cubes and pyramids, and stick to the models we have almost constantly before our eyes—the sun and the moon, and when we look through a telescope, the planets; when we are thus reasonable we have a choice between three shapes.

One of these shapes is the lemon shape, spinning round the long axis down the middle; this shape is called a prolate spheroid (I). If we cut across it at the equator, or anywhere parallel to the equator, the section is a circle. But if we cut through a meridian (*ABC*) the section is an ellipse with the axis of rotation (*AC*) as the major axis. Also, if we mark the spheroid with degrees of latitude from the equator to the poles, the spaces between the parallels decrease toward the poles; that is, the length of a degree of latitude decreases toward the poles.



Another possible shape is the orange shape, spinning round on its short axis; this shape is called an oblate spheroid (II). Again, if we cut through parallel to the equator, the sections are circles; but if we cut through a meridian (*ABC*) the section is an ellipse with the axis of rotation (*AC*) as its minor axis. Degrees of latitude are longer toward the poles than near the equator.

The third possible shape is an ellipsoid (III). This shape differs from the spheroids in having sections parallel to the equator elliptical, and not circular; so that there are two unequal axes, or radii (OA and OB), at the equator. A section through a meridian would also be an ellipse, with OC as one of the axes.

The first choice was between the spheroids. After all, the earth is rotating, and we can form a spheroid by rotating an ellipse round one of its axes; and it is highly probable that rotation will spread matter evenly round the axis, that is, in circles, and not in ellipses. Newton and Huyghens favoured the oblate spheroid, partly because of the behaviour of pendulums that were taken to the tropics, and partly because rotation would swing matter outward at the equator in the days when the earth was plastic.

There seems to have been general agreement about the oblate spheroid; it was a reasonable idea. And then, about 1700, the Cassinis, father and son, measured an arc of the meridian through Paris. The measurement showed that a degree in the northern part was shorter than a degree in the southern part. It seemed that the earth was a prolate spheroid. Sensation. The scientific world seethed with excitement. Prolate or oblate? The question was argued with all the bitterness that mathematical disputes seem to engender. The Parisian Academy of Sciences, always susceptible to romantic excitement, hastily despatched two scientific expeditions to measure arcs of meridians, one in the far north, and the other near the equator. The original measurements that started the trouble took thirty-four years to complete, and were finished in 1718. The expeditions started in 1735; so that there was a mere sixteen or seventeen years of argument before the expeditions were sent off.

The northern expedition made their measurement through the forests of Lapland. They had the advantage of the frozen river Tornea as a level surface for their base line; and they had done with the line before a thaw came. They carried their triangulation north and south from the base line, and succeeded in measuring very nearly a degree of latitude. Except that they were frost-bitten in winter, and eaten up by flies and mosquitoes in summer, this expedition had a comparatively easy time. They returned with their measurement, the length of a degree of latitude near the Polar Circle, after sixteen months.

The equatorial expedition went to Quito in Peru, and there they stayed for ten years. They measured an arc of three degrees, over mountain and valley, with one observation post down in the valley, and the next post perhaps half a mile higher up on a mountain slope. Some of the instruments turned out to be unreliable and had to be readjusted and improved. The natives stared at the scientists in amazement and then in alarm; they had no doubt that something malign was intended toward them, perhaps the filching of their lands; telescopes can look uncomfortably like some new kind of firearm. The natives became openly hostile, and attempted to slaughter the scientists. The scientists fought them off, and continued to measure. As if that were not enough, internecine conflict broke out between the two leaders of the expedition. They would not even measure together. Each went haughtily his own way, made his own measurements, and wrote his own book about the expedition.

It is worthy of mention that the measurements were made in toises. Now there were various toises of different lengths, but after that great mathematical adventure—for all the world like one of Jules Verne's stories—a toise was emphatically 'the toise of Peru', which is 6 pieds, or 2·1315 yards.

After that, who shall say that mathematicians are not romantic?

I almost forgot to say that the result of the two expeditions was to show that a degree of latitude near the Arctic Circle is definitely longer than a degree at the equator. The oblate spheroid had it.

And after all the excitement the line measured by the Cassinis was remeasured with more care, and it turned out that the Cassinis were wrong. Why was not that done at first? you may ask. As I have shown, the answer lies in the excitable romantic natures of mathematicians, and especially of French mathematicians. They carried out the job in the grand manner.

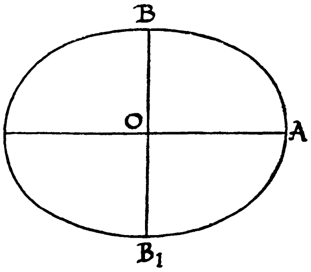
More commonplace, but never quite commonplace, surveys have since been made. Even in the small compass of the British Isles there was room for a measurement that embraced Snowdon in Wales, Slieve Donard in Ireland, and Sca Fell in Cumberland as the apices of a single triangle, each side of which is over a hundred miles long.

Surveys in all parts of the world amply confirm that the earth is an oblate spheroid. Measurements of arcs of meridians fit the equation:

$$\frac{x^2}{3963{\cdot}3^2}+\frac{y^2}{3950^2}=1,$$

which is the equation of an ellipse. That is, the equation which best satisfies the various measurements. It shows that a section through a meridian is an ellipse whose major and minor axes are 3963·3 miles and 3950 miles; the denominators are found in miles. The idea of an ellipsoid has not been neglected; but if the equatorial section is an ellipse, it is so nearly circular as to make little difference.

Two measurements are all that we need to describe an oblate spheroid completely. If we know the lengths of the semi-axes OA and OB we can construct the ellipse, which is a section through a meridian. We have only to rotate the ellipse about the minor axis BB_1 to complete the spheroid. The lengths of the axes cannot of course be measured directly. They have to be calculated from the lengths of the measured meridian arcs, and the lengths of parallels of latitude.



However, the shape of the earth is summed up in the two measurements:

polar radius: 3950 miles,
equatorial radius: $3963\frac{1}{3}$ miles.

The difference caused by the polar flattening is $13\frac{1}{3}$ miles; an observer at sea-level at one of the poles would be $13\frac{1}{3}$ miles nearer the centre of the earth than a similar observer at the equator. We are interested not so much in the actual difference in miles, as in what fraction this is of the equatorial radius. It is this fraction that determines how much the shape of the earth differs from a sphere. The fraction is called the *ellipticity*.

Ellipticity of the earth $=\frac{13\frac{1}{3}}{3963\frac{1}{3}}=\frac{1}{297}$, and that is a very little more than $\frac{1}{300}$, which is a third of one per cent.