Cambridge University Press
978-1-107-44453-9 - Numbers and Functions: Steps into Analysis: Third Edition
R. P. Burn
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NUMBERS AND FUNCTIONS

STEPS INTO ANALYSIS

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R. P. BURN

NUMBERS AND FUNCTIONS

STEPS INTO ANALYSIS

Third edition

CAMBRIDGE
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University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

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www.cambridge.org
Information on this title: www.cambridge.org/9781107444539

First published 1992
Second edition 2000
Reprinted with corrections 2006
Third edition 2015

Printed in the United Kingdom by Clays, St Ives plc

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging-in-Publication Data
Burn, R. P.
Numbers and functions : steps into analysis / R.P. Burn. – Third edition.
pages cm
Includes bibliographical references and index.
ISBN 978-1-107-44453-9
1. Mathematical analysis. I. Title.
QA300.B88 2015
515–dc23
2014046495

ISBN 978-1-107-44453-9 Paperback

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APPENDICES

*Preface to the first edition**

This text is written for those who have studied calculus in the sixth form at school, and are now ready to review that mathematics rigorously and to seek precision in its formulation. The question sequence given here tackles the key concepts and ideas one by one, and invites a self-imposed precision in each area. At the successful conclusion of the course, a student will have a view of the calculus which is in accord with modern standards of rigour, and a sound springboard from which to study metric spaces and point set topology, or multi-dimensional calculus.

Generations of students have found the study of the foundations of the calculus an uncomfortable business. The reasons for this discomfort are manifold.

(1) The student coming from the sixth form to university is already familiar with Newtonian calculus and has developed confidence in the subject by using it, and experiencing its power. Its validity has been established for him/her by reasonable argument and confirmed by its effectiveness. It is not a source of student uncertainty and this means that an axiomatic and rigorous presentation seems to make heavy weather of something which is believed to be sound, and criticisms of Newtonian calculus seem to be an irritating piece of intellectual nit-picking.

(2) At an age when a student's critical capacity is at its height, an axiomatic presentation can have a take-it-or-leave-it quality which *feels* humiliating: axioms for the real numbers have none of the 'let's-play-a-game' character by which some simpler systems appeal to the widespread interest in puzzles. The bald statement of axioms for the real numbers covers up a significant process of decision-making in their choice, and the Axiom of Completeness, which lies at the heart of most of the main results in analysis, seems superfluous at first sight, in whatever form it is expressed.

* The text of this preface was revised slightly for the second edition.

(3) Even when the axiom system has been accepted, proofs by contradiction can be a stumbling block, either because the results are unbelievable, as in the case of irrationality or uncountability, or because they make heavy weather of such seemingly obvious results as the theorem that a convergent sequence cannot have two limits.
(4) Definitions, particularly those of limits and continuity, appear strangely contrived and counter-intuitive.

There are also discomforts of lesser moment which none the less make the subject indigestible:

(5) the abstract definition of a function (when most students have only used the word function to mean a formula),
(6) the persistent use of inequalities in argument to tame infinity and infinitesimals,
(7) and proofs by induction (which play an incidental rôle in most school courses).
(8) The student who has overcome these hurdles will find that some of the best textbooks will present him/her with exercises at the end of each chapter which are so substantial that it could be a term's work to complete even those associated with three hours of lectures.
(9) The student who seeks help in the bibliography of his/her current text may find that the recommended literature is mostly for 'further reading'.

Most of these difficulties are well-attested in the literature on mathematical education. (See for example articles published in *Educational Studies in Mathematics* throughout the 1980s or the review article by David Tall (1992).)

With these difficulties in mind we may wonder how any students have survived such a course! The questions which they ask analysis lecturers may reveal their methods. Although I believed, as an undergraduate, that I was doing my best, I remember habitually asking questions about the details of the lecturer's exposition and never asking about the main ideas and results. I realise now that I was exercising only secretarial skills in the lecture room, and was not involved in the overall argument. A more participatory style of learning would have helped. Many of those who have just completed a degree in mathematics will affirm that 0.9 recurring is not the limit of a sequence, but an ordinary number less than 1! This is not evidence of any lack of intelligence: through the nineteenth century the best mathematicians stumbled because of the difficulty of imagining dense but incomplete sets of points and the seeming unreality of continuous but non-differentiable functions. There was real discomfort too in banning the language of infinitesimals from the discussion of limits. In their difficulties today, students have much in common with the best mathematicians of the nineteenth century.

It has been said that the most serious deficiency in undergraduate mathematics is the lack of an existence theorem for undergraduates! Put in other words, by an eminent educationalist, 'If I had to reduce all of educational psychology to just one principle, I would say this: the most important single factor influencing learning is what the learner already knows. Ascertain this, and teach him accordingly.' (Ausubel, 1968). There is a degree of recognition of this principle in virtually every elementary text on analysis, when, in the exercises at the end of a chapter, the strictly logical order of presentation is put aside and future results anticipated in order that the student should better understand the points at issue. The irrationality of π may be presumed before the number itself has been defined. The trigonometric, exponential and logarithmic functions almost invariably appear in exercises before they have been formally or analytically defined. In this book, my first concern, given the subject matter, has been to let students *use what they already know* to generate new concepts, and to explore situations which invite new definitions. In working through each chapter of the book the student will come to formulate, in a manner which respects modern standards of rigour, part of what is now the classical presentation of analysis. The formal achievements of each chapter are listed in summaries, but on the way there is no reluctance to use notions which will be familiar to students from their work in the sixth form. This, after all, is the way the subject developed historically. Sixth-form calculus operates with the standards of rigour which were current in the middle of the eighteenth century and it was from such a standpoint that the modern rigorous analysis of Bolzano, Cauchy, Riemann, Weierstress, Dedekind and Cantor grew.

So the first principle upon which this text has been constructed is that of involving the student in the generation of new concepts by using ideas and techniques which are already familiar. The second principle is that generalisation is one of the least difficult of the new notions which a student meets in university mathematics. The judiciously chosen special case which may be calculated or computed, provides the basis for a student's own formulation of a general theorem and this sequence of development (from special case to general theorem) keeps the student's understanding active when the formulation of a general theorem on its own would be opaque.

The third principle, on which the second is partly based, is that every student will have a pocket calculator with 'scientific' keys, and access to graph drawing facilities on a computer. A programmable calculator with graphic display will possess all the required facilities.

There is a fourth principle, which could perhaps be better called an ongoing tension for the teacher, of weighing the powerful definition and consequent easy theorem on the one hand, against the weak definition (which seems more meaningful) followed by the difficult theorem on the other. Which is the better teaching strategy? There is no absolute rule here. However

I have chosen 'every infinite decimal is convergent' as the axiom of completeness, and then established the convergence of monotonic bounded sequences adapting an argument given by P. du Bois-Reymond in 1882, and used by W. F. Osgood, 1907. Certainly to assume that monotonic bounded sequences are convergent and to deduce the convergence of the sequence for an infinite decimal takes less paper than the converse, but I believe that, more often than has usually been allowed, the combination of weaker definition and harder theorem keep the student's feet on the ground and his/her comprehension active.

Every mathematician knows theorems in which propositions are proved to be equivalent but in which one implication is established more easily than the other. A case in point is the neighbourhood definition of continuity compared with the convergent sequence definition. After considerable experience with both definitions it is clear to me that the convergent sequence definition provides a more effective teaching strategy, though it is arguable that this is only gained by covert use of the Axiom of Choice. (A detailed comparison of different limit definitions from the point of view of the learner is given in ch. 14 of Hauchart and Rouche.) As I said earlier, the issue is not one of principle, but simply an acknowledgement that the neat piece of logic which shortens a proof *may* make that proof and the result *less* comprehensible to a beginner. More research on optimal teaching strategies is needed.

It sometimes seems that those with pedagogical concerns are soft on mathematics. I hope that this book will contradict this impression. If anything, there is more insistence here than is usual that a student be aware of which parts of the axiomatic basis of the subject are needed at which juncture in the treatment.

The first two chapters are intended to enable the student who needs them to improve his/her technique and his/her confidence in two aspects of mathematics which need to become second nature for anyone studying university mathematics. The two areas are those of mathematical induction and of inequalities. Ironically, perhaps, in view of what I have written above, the majority of questions in chapter 2 develop rudimentary properties of the number system from stated axioms. These questions happen to be the most effective learning sequences I know for generating student skills in these areas. My debt to Landau's *Foundations of Analysis* and to Thurston's *The Number-system* will be evident. There is another skill which these exercises will foster, namely that of distinguishing between what is familiar and what has been proved. This is perhaps the key distinction to be drawn in the transition from school to university mathematics, where it is expected that everything which is to be assumed without proof is to be overtly stated. Commonly, a first course in analysis contains the postulational basis for most of a degree course in mathematics (see appendix 1). This justifies lecturers being particularly fussy about the reasoning used in proofs in analysis. In the second chapter we also establish various classical inequalities to use in later work on convergence.

In the third chapter we step into infinite processes and define the convergence of sequences. It is the definition of limit which is conventionally thought to be the greatest hurdle in starting analysis, and we define limits first in the context of null sequences, the preferred context in the treatments of Knopp (1928) and Burkill (1960). In order to establish basic theorems on convergence we assume Archimedean order. When the least upper bound postulate is used as a completeness axiom, it is common to deduce Archimedean order from this postulate. Unlike most properties established from a completeness axiom, Archimedean order holds for the rational numbers, and indeed for any subfield of the real numbers. So this proof can mislead. The distinctive function of Archimedean order in banishing infinite numbers and infinitesimals, whether the field is complete or not, is often missed. The Archimedean axiom expressly forbids ∞ being a member of the number field. Now that non-standard analysis is a live option, clarity is needed at this point. In any case, the notion of completeness is such a hurdle to students that there is good reason for proving as much as possible without it. We carry this idea through the book by studying sequences *without* completeness in chapter 3 and *with* completeness in chapters 4 and 5; continuous functions and limits *without* completeness in chapter 6 and *with* completeness in chapter 7; differentiation *without* completeness in chapter 8 and *with* completeness in chapter 9.

The fourth chapter is about the completeness of the real numbers. We identify irrational numbers and contrast the countability of the rationals with the uncountability of the set of infinite decimals. We adopt as a completeness axiom the property that every infinite decimal is convergent. We deduce that bounded monotonic sequences are convergent, and thereafter standard results follow one by one. Of the possible axioms for completeness, this is the only one which relates directly to the previous experience of the students. With completeness under our belt, we are ready to tackle the convergence of series in chapter 5.

The remaining chapters of the book are about real functions and start with a section which shows why the consideration of limiting processes requires the modern definition of a function and why formulae do not provide a sufficiently rich diet of possibilities. By adopting Cantor's sequential definition of continuity, a broad spectrum of results on continuous functions follows as a straightforward consequence of theorems about limits of sequences. The second half of chapter 6 is devoted to reconciling the sequential definitions of continuity and limit with Weierstrass' neighbourhood definitions, and deals with both one- and two-sided limits. These have been placed as far on in the course as possible. There is a covert appeal to the

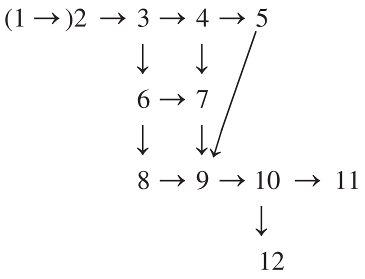
Axiom of Choice in the harder proofs. Chapter 6 builds on chapter 3, but does not depend on completeness in any way. In chapter 7 we establish the difficult theorems about continuity on intervals, all of which depend on completeness. Taking advantage of the sequential definition of continuity, we use completeness in the proofs by claiming that a bounded sequence contains a convergent subsequence. There is again a covert appeal to the Axiom of Choice.

Chapters 8 and 9, on differentiation, are conventional in content, except in stressing the distinction between those properties which do not depend on completeness, in chapter 8 (the definition of derivative and the product, quotient and chain rules), and those which do, in chapter 9 (Rolle's Theorem to Taylor's Theorem). The differentiation of inverse functions appears out of place, in chapter 8. Chapter 10, on integration, starts with the computation of areas in ways which were, or could have been, used before Newton, and proceeds from these examples to the effective use of step functions in the theory of the Riemann integral. Completeness is used in the definition of upper and lower integrals.

The convention of defining logarithmic, exponential and circular functions either by neatly chosen integrals or by power series is almost universal. This seems to me to be an excellent procedure in a *second* course. But the origins of exponentials and logarithms lie in the use of indices and that is the starting point of our development in chapter 11. Likewise our development of circular functions starts by investigating the length of arc of a circle. The treatment is necessarily more lengthy than is usual, but offers some powerful applications of the theorems of chapters 1 to 10.

A chapter on uniform convergence completes the book. Some courses, with good reason, postpone such material to the second year. This chapter rounds off the problem sequence in two senses: firstly, by the discussion of term-by-term integration and differentiation, it completes a university-style treatment of sixth-form calculus; and secondly by discussing the convergence of functions it is possible to see the kind of questions which provoked the rigorous analysis of the late nineteenth century.

The interdependence of chapters is illustrated below.



One review of my *Pathway into Number Theory* (*Times Higher Educational Supplement*, 3.12.82) suggested that a pathway to analysis would be of more value than a pathway to number theory. While not disagreeing with the reviewer, I could not, and still today cannot, see the two tasks as comparable. The subtlety of the concepts and definitions of undergraduate analysis is of a different order. But the reviewer's challenge has remained with me, and the success I have seen students achieve with my pathways to number theory and geometrical groups has spurred me on. None the less, I offer these steps (notice the cautious claim by comparison with the earlier books) aware that they contain more reversals of what I regard as an optimal teaching sequence than the earlier pathways.

It may be helpful to clarify the differences between the present text and other books on analysis which have the word 'Problem' in their titles. I refer firstly to the books of the Schaum series. Although their titles read *Theory and problems of* . . . the books consist for the most part of solutions. Secondly, a book with the title *Introductory Problem Course in Analysis and Topology* written by E. E. Moise consists of a list of theorems cited in logical sequence. Thirdly, the book *Problems and Propositions in Analysis* by G. Klambauer provides an enriching supplement to any analysis course, and, in my opinion, no lecturer in the subject should be without a copy. And finally there is the doyen of all problem books, that by Pólya and Szegö (1976), which expects greater maturity than the present text but, again, is a book no analysis lecturer should be without.

Sometimes authors of mathematics books claim that their publications are 'self-contained'. This is a coded claim which may be helpful to an experienced lecturer, but can be misleading to an undergraduate. It is never true that a book of university mathematics can be understood without experience of other mathematics. And when the concepts to be studied are counter-intuitive, or the proofs tricky, it is not just that one presentation is better than another, but that all presentations are problematic, and that whichever presentation a student meets *second* is more likely to be understood than the one met *first*. For these reasons I persistently encourage the consultation of other treatments. While it is highly desirable to recommend a priority course book (lest the student be entirely at the mercy of the lecturer) the lecturer needs to use the ideas of others to stimulate and improve his/her own teaching; and, because students are different from one another, no one lecturer or book is likely to supply quite what the student needs.

There is a distinction of nomenclature of books on this subject, with North American books tending to include the word 'calculus' in their titles and British books the word 'analysis'. In the nineteenth century the outstanding series of books by A. L. Cauchy clarified the distinction. His first volume was about convergence and continuity and entitled 'Course of Analysis' and his later volumes were on the differential and integral calculus.

The general study of convergence thus preceded its application in the context of differentiation (where limits are not reached) and in integration (where the limiting process occurs by the refinement of subdivisions of an interval). British university students have met a tool-kit approach to calculus at school and the change in name probably assists the change in attitude required in transferring from school to university. But the change of name is unhelpful to the extent that it is part of the purpose of every first course of analysis in British universities to clarify the concepts of school calculus and to put familiar results involving differentiation and integration on a more rigorous foundation.

I must express gratitude and indebtedness to many colleagues: to my own teacher Dr J. C. Burkill for his pursuit of simplicity and clarity of exposition; to Hilary Shuard my colleague at Homerton College whose capacity to put her finger on a difficulty and keep it there would put a terrier to shame (an unpublished text on analysis which she wrote holds an honoured place in my filing cabinet; time and again, when I have found all the standard texts unhelpful, Hilary has identified the dark place, and shown how to sweep away the cobwebs); to Alan Beardon for discussions about the subject over many years; to Dr D. J. H. Garling for redeeming my mistakes, and for invaluable advice on substantial points; to Dr T. W. Körner for clarifying the relationship between differentiability and invertibility of functions for me by constructing a differentiable bijection $\mathbb{Q} \to \mathbb{Q}$ whose inverse is not continuous; and to Dr F. Smithies for help with many historical points. At a fairly late stage in the book's production I received a mass of detailed and pertinent advice from Dr Tony Gardiner of Birmingham University which has led to many improvements. This book would not yet be finished but for the sustained encouragement and support of Cambridge University Press. The first book which made me believe that a humane approach to analysis might be possible was *The Calculus, a Genetic Approach* by O. Toeplitz. The historical key to the subject which he picked up can open more doors than we have yet seen. I still look forward to the publication of an undergraduate analysis book, structured by the historical development of the subject during the nineteenth century. [Added in 1995.] David Bressoud's new book is greatly to be welcomed, but we still await a text inspired by the history surrounding the insights of Weierstrass, and especially the development of the notion of completeness.

I would like to be told of any mistakes which students or lecturers find in the book.

School of Education, R. P. Burn
Exeter University, EX1 2LU
January 1991

Preface to the second edition

Under the inspiration of David Fowler, driven by the leadership of David Epstein, eased by the teaching ideas of Alyson Stibbard and challenged by the research of Lara Alcock, a remarkable transformation of the way in which students begin to study analysis took place at Warwick University. The division of teaching time between lectures and problem-solving by students in class changed to give student problem-solving pride of place. Many of the problems the students tackled in this approach to analysis were taken from the first five chapters of the first edition of this book. The experience has suggested a number of improvements to the original text. This new edition is intended to embody these improvements. The most obvious is the playing down of the Peano postulates for the natural numbers and algebraic axioms for the real numbers, which affects chapter 1 and the beginning of chapter 2. The second is in the display of 'summaries', which now appear when a major idea has been rounded off, not just at the end of the chapter. The third is an increase in the number of diagrams, and the fourth is the introduction of simple (and I hope evocative) names for small theorems, so that they may be cited more readily than when they only have a numerical reference. There are also numerous smaller points not just about individual questions. In particular, there is a suggestion of two column proofs in establishing the rudimentary properties of inequalities in chapter 2, and least upper bounds now figure more substantially in chapter 4. The stimulus of discussions with David Epstein both about overall strategy and about the details of question after question has been a rare and enriching experience.

I must also thank Paddy Paddam, previously of Cambridge, who has worked through every question in the text, and made invaluable comments to me from a student's eye view. I have also taken the opportunity to correct and improve the historical references and notes.

Kristiansand
March 2000

Preface to the third edition

In addition to taking the opportunity to correct and improve details in the text and historical notes, this edition incorporates two new ideas. In chapter 3, the introduction to limits, especially in relation to null sequences, builds on the method of Archimedes for finding areas and volumes, and the early questions in chapter 10 investigate the determination of areas bounded by continuous monotonic curves without an appeal to completeness. This extends the structure of the book from limits, continuity and differentiation with and without completeness to consider integration also, with and without completeness. As in so many places in previous editions, the clarifying insights have emerged from seeking to understand the historical development. All the changes have been made in such a way as to maintain, so far as possible, the sequence and numbering of questions in the second edition. Only questions 3.24–26, 28 and 10.1, 3–9 have been substantially changed.

I gratefully acknowledge discussion of historical matters with Jeremy Gray, and also prolonged dialogue with Dirk Nelson which has improved the clarity and accuracy of the text.

Exeter
May 2014

Glossary

qn 27 refers to question 27 of the same chapter.
qn 6.27 refers to question 27 of chapter 6.
6.27^+ refers to what immediately follows qn 6.27.
6.27^- refers to what immediately precedes qn 6.27.

$x \in A$	x is an element of the set A; for example, $x \in \{x, \ldots\}$
$x \notin A$	x is not an element of the set A
$\{x \mid x \in A\}$ or $\{x : x \in A\}$	the set A
$A \subseteq B$	A is a subset of B every member of A is a member of B $x \in A \Rightarrow x \in B$
$A \cup B$	the union of A and B $\{x \mid x \in A \text{ or } x \in B\}$
$A \cap B$	the intersection of A and B $\{x \mid x \in A \text{ and } x \in B\}$
$A \backslash B$	$\{x \mid x \in A \text{ and } x \notin B\}$
$A \times B$	the cartesian product $\{(a, b) \mid a \in A, b \in B\}$
$f : A \to B$	the function f, a subset of $A \times B$, $\{(a, f(a)) \mid a \in A, f(a) \in B\}$ A is the domain and B is the co-domain of the function f
$f(A)$	$\{f(x) \mid x \in A\} \subseteq B$
$x \mapsto f(x)$	the function f, x is an element of the domain of f
$\mathbb{N}$	the set of natural numbers or counting numbers $\{1, 2, 3, \ldots\}$

$\mathbb{Z}$	the set of integers $\{0, \pm 1, \pm 2, \ldots\}$
$\mathbb{Z}^+$	the set of positive integers $\{1, 2, 3, \ldots\}$
$\mathbb{Q}$	the set of rational numbers $\{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}\backslash\{0\}\}$
$\mathbb{Q}^+$	the set of positive rationals $\{x \mid x \in \mathbb{Q}, 0 < x\}$
$\mathbb{R}$	the set of real numbers, or infinite decimals
$\mathbb{R}^+$	the set of positive real numbers $\{x \mid x \in \mathbb{R}, 0 < x\}$
$\lfloor x \rfloor$	the integral part of x, also called floor x $\lfloor x \rfloor \leqslant x < \lfloor x \rfloor + 1$
$[a, b]$	closed interval $\{x \mid a \leqslant x \leqslant b, x, a, b \in \mathbb{R}\}$
(a, b)	open interval $\{x \mid a < x < b, x, a, b \in \mathbb{R}\}$, the symbol may also denote the ordered pair, (a, b), or the two coordinates of a point in $\mathbb{R}^2$
$[a, b)$	half-open interval $\{x \mid a \leqslant x < b, x, a, b \in \mathbb{R}\}$
$[a, \infty)$	closed half-ray $\{x \mid a \leqslant x, a, x \in \mathbb{R}\}$
(a, ∞)	open half-ray $\{x \mid a < x, a, x \in \mathbb{R}\}$
(a_n)	the sequence $\{a_n \mid n \in \mathbb{N}, a_n \in \mathbb{R}\}$ that is, a function: $\mathbb{N} \to \mathbb{R}$
$(a_n) \to a$ as $n \to \infty$	For any $\varepsilon > 0$, $\lvert a_n - a \rvert < \varepsilon$ for all $n > N$
$\binom{n}{r}$	$\dfrac{n!}{(n-r)!r!}$ when $n \in \mathbb{Z}^+$
$\binom{a}{r}$	$\dfrac{a(a-1)(a-2)\ldots(a-r+1)}{r!}$ the binomial coefficient when $a \in \mathbb{R}$
$\sum_{n=1}^{n=N} a_n$	$a_1 + a_2 + a_3 + \ldots + a_N$
$\ln x$	the natural logarithm of x $\log_e x$