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R. P. Burn
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PART I

Numbers

1

Mathematical induction

Preliminary reading: Rosenbaum, Polya ch. 7.
Concurrent reading: Sominskii.
Further reading: Thurston chs 1 and 2, Ledermann and Weir.

The set of all whole numbers $\{1, 2, 3, \ldots\}$ is often denoted by $\mathbb{N}$. We will usually call $\mathbb{N}$ the set of *natural numbers*, and, sometimes, the set of *positive integers*. Note that $\mathbb{N}$ excludes 0.

1 $1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}.$

Is this proposition true or false?
Test it when $n = 1$, $n = 2$ and $n = 3$.
How many values of n should you test if you want to be sure it is true for all natural numbers n?

If we write $f(n) = \dfrac{n(n+1)(2n+1)}{6}$, show that

$$f(n) + (n+1)^2 = f(n+1).$$

Now suppose that the proposition with which we started holds for some particular value of n: add $(n+1)^2$ to both sides of the equation and deduce that the proposition holds for the next value of n.
Since we have already established that the proposition holds for $n = 1, 2$, and 3, the argument we have just formulated shows that it must hold for $n = 4$, and then by the same argument for $n = 5$, and so on.

2 When n is a natural number, $6^n - 5n + 4$ is divisible by 5.
Check this proposition for $n = 1, 2$ and 3.
By examining the difference between this number and $6^{n+1} - 5(n+1) + 4$, show that if the proposition holds for one value of n, it holds for the succeeding value of n. When you have done this you have

established the two components of the proof of the proposition by mathematical induction.

3 Prove the following propositions by induction (some have easy alternative proofs which do not use induction):

(i) (*Triangular numbers*) $1+2+3+\ldots+n=\frac{1}{2}n(n+1)$,
(ii) $a+(a+d)+(a+2d)+\ldots+(a+(n-1)d)=\frac{1}{2}n(2a+(n-1)d)$,
(iii) $1^3+2^3+3^3+\ldots+n^3=[\frac{1}{2}n(n+1)]^2$,
(iv) $1\cdot 2+2\cdot 3+3\cdot 4+\ldots+n(n+1)=\frac{1}{3}n(n+1)(n+2)$,
(v) $1+2+4+\ldots+2^{n-1}=2^n-1$,
(vi) $1+x+x^2+\ldots+x^{n-1}=\dfrac{x^n-1}{x-1}$, provided $x\neq 1$.
(vii) (optional) $1+2x+3x^2+\ldots+nx^{n-1}=\dfrac{nx^n}{x-1}-\dfrac{x^n-1}{(x-1)^2}$, provided $x\neq 1$.

4 Pascal's triangle, shown here, is defined inductively, each entry being the sum of the two (or one) entries in the preceding row nearest to the new entry. The $(r+1)$th entry in the nth row is denoted by $\dbinom{n}{r}$

$$\begin{array}{ccccccccccc} & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ 1 & \ldots & & \ldots & & \ldots & & \ldots & & \ldots & 1 \end{array}$$

and called 'n choose r' because it happens to count the number of ways of choosing r objects from a set of n objects. Notice that we always have $0\leq r\leq n$. From the portion of Pascal's triangle which has been shown, for example,

$$\binom{3}{0}=1,\ \binom{3}{1}=3,\ \binom{3}{2}=3 \text{ and } \binom{3}{3}=1.$$

From the definition of Pascal's triangle we have

$$\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r},\quad \text{when } 0<r<n.$$

Prove by induction that $\dbinom{n}{r}=\dfrac{n!}{r!(n-r)!}$, taking $0!=1$.

Verify that this formula gives $\dbinom{n}{r}=\dbinom{n}{n-r}$.

What aspect of Pascal's triangle does this reflect?

5 *The Binomial Theorem for positive integral index*
Prove by induction that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{r}x^r + \ldots + \binom{n}{n}x^n.$$

6 Examine each of the propositions:

(i) $2^{n-1} \le n^2$,
(ii) $n^2 + n + 41$ is a prime number.

Find values of n for which these propositions hold and also a value of n for which each is false.
If you were to attempt a proof by induction of either of these propositions, where would the proof break down?

7 Examine the numbers $6^n - 5n + 1$ and $6^{n+1} - 5(n+1) + 1$.
Find their difference.
Deduce that if the first of these numbers were divisible by 5 then the second would also be divisible by 5. Deduce also that if the second were divisible by 5 then the first would also be divisible by 5.
Is the first number divisible by 5 when $n = 1$?
Are there any values of n for which these numbers are divisible by 5?
If you were to attempt a proof by induction that the first number was divisible by 5, where would the proof break down?

8 Define $y^{(0)} = y$, $y^{(1)} = \dfrac{dy}{dx}$, and $y^{(n+1)} = \dfrac{dy^{(n)}}{dx}$.

(i) Let $y = \ln(1+x)$, so $\dfrac{dy}{dx} = \dfrac{1}{1+x}$.
Prove that for $n \ge 1$, $(1+x)y^{(n+1)} + ny^{(n)} = 0$.
Deduce that when $x = 0$, $y^{(n+1)} = (-1)^n n!$

(ii) Let $y = \arctan x$, so $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$.
Prove that for $n \ge 1$,
$(1+x^2)y^{(n+2)} + 2(n+1)xy^{(n+1)} + n(n+1)y^{(n)} = 0$.
Deduce that when $x = 0$, $y^{(2n)} = 0$ and $y^{(2n+1)} = (-1)^n(2n)!$

There is ambiguity in the literature on the question of whether $\mathbb{N}$ contains 0 or not. If mathematics starts with sets, and particularly the empty set, the equipment for counting the elements of finite sets must include 0. The psychological origins of counting, however, start with 1. It is this convention we have called *natural*. Both conventions are well-established.

THE PRINCIPLE OF MATHEMATICAL INDUCTION

A proposition, $P(n)$, relating to a natural number n, is valid for all natural numbers n, if

(a) $P(1)$ is true; the proposition is valid for $n = 1$ and
(b) $P(n) \Rightarrow P(n+1)$; the proposition for n implies the proposition for $n + 1$.

Historical Note

The study of triangular numbers is said to go back to the Pythagoreans (6th century BC). The sum of the first n squares was known to Archimedes (c. 250 BC) and the sum of the first n cubes to the Arabs (c. AD 1010). The justification of these forms used induction implicitly.

The earliest use of proof by mathematical induction in the literature is by Maurolycus in his study of polygonal numbers published in Venice in 1575. Pascal knew the method of Maurolycus and used it for work on the binomial coefficients (c. 1657). In 1713, Jacques Bernoulli used an inductive proof to make rigorous the claim of Wallis (1656) that

$$\sum_{r=0}^{r=n} \frac{r^k}{n^{k+1}} \to \frac{1}{k+1} \text{ as } n \to \infty.$$

By the latter part of the eighteenth century, induction was being used by several authors. The name 'mathematical induction' as distinct from scientific induction is due to De Morgan (1838). The use of inductive definitions long pre-dates this method of proof.

The well-ordering principle, that every non-empty set of natural numbers has a least member, is equivalent to mathematical induction (see Ledermann and Weir, 1996). Well-ordering was used by Euclid (VII.31) to show that every integer has a prime factor and by Fermat (1637) in his proof of the non-existence of integral solutions to $x^4 + y^4 = z^2$. However, the recognition of the relationship between well-ordering and induction is recent, and from an historical point of view, awareness of the two principles developed independently.

During the nineteenth century there were several attempts to give a formal description of the natural numbers, notably by H. Grassmann (1861) and H. von Helmholtz (1887) both of whom assumed the Principle of Induction. R. Dedekind (1888) defined $\mathbb{N}$ as an arbitrary infinite chain and was able to prove the Principle of Induction. But in 1889, G. Peano, working with Dedekind's material, adopted the Principle of Mathematical Induction as one of his five postulates for defining the natural numbers, and his account remains standard to this day though Peano's Postulates admit non-standard models, which we exclude with Archimedean order, in chapter 3.

Answers and comments

1 Let $P(n)$ mean '$1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$'.
You checked that $P(1)$, $P(2)$ and $P(3)$ were all true.
The algebra you did showed that $P(n) \Rightarrow P(n+1)$.
So (by induction) $P(n)$ is true for all natural numbers n.

2 Let $P(n)$ mean '$6^n - 5n + 4$ is divisible by 5'.
Because $5|5$, $P(1)$ is true. Because $5|30$, $P(2)$ is true. Because $5|205$, $P(3)$ is true.
The difference between the expression for $n+1$ and the expression for n is $5 \cdot 6^n - 5$, which is divisible by 5, so $P(n) \Rightarrow P(n+1)$. And (by induction) $P(n)$ is true for all natural numbers n.

3 Take each of the six propositions in turn as $P(n)$. Verify the truth of $P(1)$ in each case. The algebra behind the proof of $P(n) \Rightarrow P(n+1)$ is as follows:

(i) $\frac{1}{2}n(n+1) + (n+1) = \frac{1}{2}(n+1)(n+2)$,
(ii) $\frac{1}{2}n[2a + (n-1)d] + (a + nd) = \frac{1}{2}(n+1)(2a + nd)$,
(iii) $[\frac{1}{2}n(n+1)]^2 + (n+1)^3 = [\frac{1}{2}(n+1)(n+2)]^2$,
(iv) $\frac{1}{3}n(n+1)(n+2) + (n+1)(n+2) = \frac{1}{3}(n+1)(n+2)(n+3)$,
(v) $(2^n - 1) + 2^n = 2^{n+1} - 1$,
(vi) $\dfrac{x^n - 1}{x - 1} + x^n = \dfrac{x^{n+1} - 1}{x - 1}$,
(vii) $$\frac{nx^n}{x-1} - \frac{x^n - 1}{(x-1)^2} + (n+1)x^n = \frac{nx^n + (n+1)x^n(x-1)}{x-1} - \frac{x^n - 1}{(x-1)^2}$$
$$= \frac{(n+1)x^{n+1}}{x-1} - \frac{x^{n+1} - 1}{(x-1)^2}.$$

4 If $P(n)$ is the proposition we are asked to prove for $0 \le r \le n$,
$$\binom{n}{0} = \binom{n}{n} = 1$$
is easily checked, and incorporates the truth of $P(1)$. If we assume $P(n)$ and apply it to the Pascal triangle property, we get $P(n+1)$.
Symmetry about a vertical axis.

5 For the inductive step, one must show that the coefficient of x^r in
$(1+x)^{n+1} = (1+x)^n(1+x)$ is $\dbinom{n}{r-1} + \dbinom{n}{r}$.

6 (i) True for $n \le 6$, false for $7 \le n$.
(ii) True for $n \le 39$, false for $n = 40$, or a multiple of 41, or $40 + k^2$.

$P(n)$ does not imply $P(n+1)$.

7 $6^n - 5n + 1$ divisible by $5 \Leftrightarrow 6^{n+1} - 5(n+1) + 1$ divisible by 5.
But both statements are false because the first statement is false when $n = 1$. $P(1)$ is false.

8 (i) $(1+x)dy/dx = 1$, or $(1+x)y^{(1)} = 1$, so $(1+x)y^{(2)} + y^{(1)} = 0$, and the proposition holds for $n = 1$.
Suppose $(1+x)y^{(n+1)} + ny^{(n)} = 0$, differentiating we have
$(1+x)y^{(n+2)} + (n+1)y^{(n+1)} = 0$, and the proposition for n implies the proposition for $n+1$.
When $x = 0$, $y^{(1)} = 1$ and $y^{(n+1)} = -ny^{(n)}$. So
$y^{(n+1)} = (-1)^n n! \Rightarrow y^{(n+2)} = (-1)^{n+1}(n+1)!$
and the result holds by induction.

(ii) Differentiate $(1+x^2)y^{(1)} = 1$ twice to get result for $n = 1$.
Differentiate result for n to obtain result for $n+1$. Result follows by induction.
When $x = 0$, $y^{(1)} = 1$, $y^{(2)} = 0$ and $y^{(n+2)} + n(n+1)y^{(n)} = 0$. So result holds for $n = 1$.
Also $y^{(2n)} = 0 \Rightarrow y^{(2n+2)} = 0$, and $y^{(2n+1)} = (-1)^n(2n)!$
$\Rightarrow y^{(2n+3)} = -(2n+1)(2n+2)(-1)^n(2n)! = (-1)^{n+1}(2n+2)!$
So result holds by induction.

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Inequalities

Preliminary reading: Beckenbach and Bellman.
Concurrent reading: Korovkin, Kazarinoff.
Further reading: Thurston ch. F, Ivanov ch. 4.

Positive numbers and their properties

The addition, subtraction, multiplication and division of numbers will work in this course the way they did for you in school. Their essential algebraic properties are listed in appendix 1 under the heading 'Algebraic properties of a field of numbers'. Look at this list briefly. Examine which of these algebraic properties hold for the set of integers

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\},$$

and then check that the set of rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$$

satisfies *all* these algebraic properties. There is no need to remember the list for this course. The letter $\mathbb{Z}$ is the first letter of the German word *Zahl*, meaning *number*. The letter $\mathbb{Q}$ is the first letter of *quotient*; each rational number being a quotient of integers. The letter $\mathbb{R}$, the first letter of the word *real*, is used to denote the set of real numbers, the numbers needed for measuring distances along a line. We will presume that all the algebraic properties of numbers mentioned in appendix 1 hold for the set of real numbers. The difference between $\mathbb{Q}$ and $\mathbb{R}$ will be examined in chapter 4.

It is questions of convergence and the finding of limits which characterise a course in analysis. It is necessary to use inequalities in order to define these infinite processes. The purpose of this chapter is to sharpen your awareness of inequalities so that you know how to argue with them and what you need to be careful about when you are using them in an argument.

1 To prepare for a formal treatment of inequalities, determine for what numbers x you want to claim the inequality $2x < 3x$. Check whether you expect it to hold when $x = 1$, when $x = 0$ and when $x = -1$.

FORMAL PROPERTIES OF POSITIVE NUMBERS

1. If a is a number, then *either* $a = 0$, *or* a is positive *or* $-a$ is positive, and only one of these is true. When $-a$ is positive, a is said to be negative. This property is called the *trichotomy law* because of the *three* possibilities.
2. The sum of two positive numbers is positive. This is also described by saying that the positive numbers are *closed under addition*.
3. The product of two positive numbers is positive. This is also described by saying that the positive numbers are *closed under multiplication*.

We introduce the inequality '$<$' with the following

DEFINITION OF 'LESS THAN'
We say $a < b$ if and only if $b - a$ is positive.

The subset of positive numbers in $\mathbb{Z}$ is denoted by $\mathbb{Z}^+$, the subset of positive numbers in $\mathbb{Q}$ is denoted by $\mathbb{Q}^+$, and the subset of positive numbers in $\mathbb{R}$ is denoted by $\mathbb{R}^+$.

We can go on from here to define $b > a$ if and only if $a < b$, and to modify these definitions for $\leq$ and $\geq$. We will keep to '$<$' until some elementary properties have been established. In the proofs, be careful only to use the properties of '$<$' which have been given, or which you have successfully deduced from them. The answers to these questions have been set out so as to highlight the reasons for each step.

2 Use the *definition of less than* to give an inequality equivalent to the proposition 'b is positive'.

3 If $0 < -a$, use the definition to prove that $a < 0$.

4 If $a < 0$, use the definition to prove that $0 < -a$.

5 Use the law of trichotomy to prove that for any number a, either $a = 0$ or $0 < a$ or $a < 0$, and only one of these is true.

6 If $-b < -a$, use the definition to prove that $a < b$.

7 *Extended trichotomy law*
Apply the trichotomy law to the number $b - a$ to prove that for any two numbers a and b, *either* $a = b$, *or* $a < b$, *or* $b < a$, and exactly one of these is true.

8 If $a < b$, use the definition to prove $a + c < b + c$.