ABERRATION

AND

THE ELECTROMAGNETIC FIELD.

ABERRATION

AND SOME OTHER PROBLEMS
CONNECTED WITH

THE ELECTROMAGNETIC FIELD.

ONE OF TWO ESSAYS TO WHICH THE ADAMS PRIZE WAS AWARDED IN 1899, IN THE UNIVERSITY OF CAMBRIDGE.

BY

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PREFACE.

THE subject selected by the Examiners for the Adams Prize for 1899 was

THE THEORY OF THE ABERRATION OF LIGHT.

The phenomena of aberration depend upon the relations of the ether and matter and must therefore be intimately associated with many other facts of nature. In the following essay an attempt is made to construct a theory of the electromagnetic field which shall be consistent with the modern interpretation of chemical, optical and magnetic phenomena in terms of electrically charged particles*. That a particular case of such a theory would explain aberration was proved by H. A. Lorentz in the year 1892†.

Since the essay was returned by the examiners I have thought it advisable to remove some defects and obscurities which did not appreciably affect the main argument. It has been chiefly in the latter portion of Part III. that such changes

* A short historical account of electrochemical theories has been given by Richarz (*Phil. Mag.* 39, p. 529, 1895).

† La théorie électromagnétique de Maxwell et son application aux corps mouvants. (*Archives néerlandaises des Sciences exactes et naturelles*, T. xxv.: also published separately by E. J. Brill, Leyden, 1892.)

have been made: and the five pages from (λ), § 56 (p. 78) to the end of Part III. (p. 83) are recent additions for the sake of which the publication has been delayed. I realised last autumn that many of Quincke's classical experiments with magnetic media, on which I had relied for verification, were capable of an interpretation more in accordance with the older theories: and noticing that his investigation of compression* would, under certain conditions, be decisive, I ventured to ask him for some particulars. He has repeated and amplified the experiment† with the conditions modified in order more completely to test the results of § 53. I here wish to express the deepest sense of gratitude to Prof. Quincke for the extremely generous manner in which he has allowed me to avail myself of his great skill.

In Part I. attention has been drawn to an inconsistency in the work of Maxwell, Helmholtz, and others. A medium when magnetised is by them initially regarded as consisting of particles with polar properties—of magnetic 'doublets,' if a hydrodynamical term be permitted. This conception of polarisation will in future be called 'molecular.' On the other hand, when the electromotive force round a moving circuit is determined, the 'induction' or 'magnetic polarisation' is regarded as continuous and completely filling space: it is the change in the number of 'tubes' or 'lines' which is estimated. A similar divergence occurs in relation to electric phenomena. We may follow Maxwell and regard electric displacement as 'continuous'; or we may adopt the point of view of Helmholtz in his paper on anomalous dispersion‡ and suppose that the polarisation of a dielectric consists in a slight disturbance of its ions.

* *Wiedemann's Annalen*, XXIV. p. 380, § 68.

† *Sitzungsber. d. Akad. d. Wiss. z. Berlin*, April 19, 1900.

‡ Elektromagnetische Theorie der Farbenzerstreuung, *Wissenschaft. Abhand.* III. pp. 505—525.

After a study (§§ 3—10) of the geometrical and kinematical properties of a molecularly polarised medium we have formed analytical expressions for Maxwell's fundamental ideas connecting the line-integrals of force round circuits moving in any manner with the fluxes through them. These are the general equations of the field; and those appropriate to molecular polarisation are different from the equations corresponding to continuous polarisation {cf. equations (22) and (27) with (13) and (17)}. The relations between **E**, **H**, the electric and magnetic forces at a point fixed in the ether, and **E**′, **H**′, the forces at a point moving with velocity **u** through it, are also different in the two theories {cf. (14) with (24) and (29)}.

Lorentz in a second paper* adopts the molecular hypothesis, but considers only those cases in which the ether is stationary, the velocity **u** of the matter relative to the ether is constant, and no electric volume-density, conduction currents or magnetic media are present. By a brilliant piece of analysis he transforms (Abschnitt v. §§ 56—59) the equations of the field to those of a stationary system, and obtains (§§ 60—63, 68, &c.) the explanation of the ordinary facts connected with aberration.

The equations of Lorentz are however not sufficiently general for application to such problems as Röntgen's spinning disc, reflection from a rotating mirror, or the determination of stresses in the field.

In Part II. we investigate (§§ 19, 20) a transformation somewhat more general than that of Lorentz already alluded to, and then discuss more closely the propagation of plane waves through a drifting medium. It is found (§§ 21—24) that a medium which is isotropic when stationary, behaves when

* *Versuch einer Theorie der electrischen und optischen Erscheinungen in bewegten Körpern*, Leyden, 1895.

moving like a uniaxal crystal whose axis is parallel to the direction of drift: the difference of velocity of the two waves is of the second order, being

$$\tfrac{1}{2}(K-1)\,\mathbf{u}^2 \sin^2\theta / K^{\frac{3}{2}} V,$$

where θ is the angle between $\mathbf{u}$ and the wave normal.

Röntgen has shown* that a magnetic field is produced by rotating an uncharged glass disc between and parallel to the plates of a charged condenser. The molecular theory yields (§§ 26, 7) a complete explanation.

In Part III. we have investigated the values of the stresses in an electromagnetic field in accordance with several possible hypotheses.

In the determinations of stress made by Maxwell, Helmholtz, Kirchhoff, and Lorberg considerations have been neglected which appear to us of some importance. If we are discussing the magnetic stresses in an electromagnetic field and suppose a displacement effected, there will be changes in the conduction currents and electric forces whose intensities will be proportional to the velocity of the displacement. The result is a rate of redistribution of energy which will be comparable with the rate at which the work is done by the stresses: and it cannot therefore be satisfactory to consider merely the changes in the magnetic potential energy of the system.

Another doubtful method is that of postulating the mechanical effects of the field and then finding stresses which will account for them. This process may be illustrated by considering a magnetostatic field containing permanent magnets but no bodies with susceptibility. If the stresses consist of tensions $\mathbf{H}^2/8\pi$ along lines of force and pressures $\mathbf{H}^2/8\pi$ at right angles to them.....(α),

* *Wied, Ann.* xxxv. p. 264, 1888: xl. p. 93, 1890.

the rate at which work is done by the stresses when the velocity at any point is (u, v, w) is, in Maxwell's notation,

$$-\frac{1}{8\pi}\int dv\,\Sigma\,\{u_x(\alpha^2-\beta^2-\gamma^2)+2u_y\alpha\beta+2u_z\alpha\gamma\}.$$

This transforms by Green's theorem to

$$\frac{1}{8\pi}\int dS\,\Sigma u\,\{l(\alpha^2-\beta^2-\gamma^2)+m\,.\,2\alpha\beta+n\,.\,2\alpha\gamma\}_1^2$$

$$+\frac{1}{4\pi}\int dv\,(u\alpha+v\beta+w\gamma)(\alpha_x+\beta_y+\gamma_z),$$

where numbers 1, 2 indicate values inside and outside the magnets. It is easily seen that the surface integral expresses the rate at which work would be done by surface-forces equal to

$$\tfrac{1}{2}(\alpha_1+\alpha_2)\,v,\quad \tfrac{1}{2}(\beta_1+\beta_2)\,v,\quad \tfrac{1}{2}(\gamma_1+\gamma_2)\,v,$$

where v is the surface density: the volume integral corresponds to an internal force $\alpha\tau$, $\beta\tau$, $\gamma\tau$, per unit volume, where τ is the volume density .....(β).

On replacing τ by $-(A_x+B_y+C_z)$ the rate of doing of work transforms to

$$-\frac{1}{8\pi}\int dS\,\{\alpha^2+\beta^2+\gamma^2\}_1^2(lu+mv+nw)$$

$$+\int dv\,\{\Sigma u\,(A\alpha_x+B\alpha_y+C\alpha_z)+\Sigma u_x A\alpha$$

$$+\tfrac{1}{2}\Sigma\,(w_y+v_z)(B\gamma+C\beta)+\tfrac{1}{2}\Sigma\,(w_y-v_z)(B\gamma-C\beta)\}.$$

We now see that the system of stresses (α) is equivalent, in its action on a rigid body, to the system of forces (β), and it is also equivalent to the following system:—

(1) a surface thrust of $-\{\mathbf{H}^2\}_1^2/8\pi$ along the outward normal,

(2) a force in the interior equal to

$$(A\alpha_x+B\alpha_y+C\alpha_z,\quad A\beta_x+B\beta_y+C\beta_z,\quad A\gamma_x+B\gamma_y+C\gamma_z)$$

(i.e. to $\mathbf{I}\nabla\,.\,\mathbf{H}$) per unit volume,

(3) tensions $A\alpha$, $B\beta$, $C\gamma$,

(4) shearing-stresses represented by $\frac{1}{2}(B\gamma + C\beta)$,

(5) couples $B\gamma - C\beta$, &c. ....(γ).

Maxwell states in § 639 of his treatise that the effect of the field is the production of the force (2) and the couples (5), but he does not consider the possibility of the tensions and shears; clearly however the forces on the two poles of a magnetic particle which produce the couples would also produce the tensions (3), and similarly the shears (4) may be interpreted.

Now the system of stresses by means of which Maxwell explains (2) and (5) is given in § 641 as

$$P_{xx} = (\alpha^2 - \beta^2 - \gamma^2)/8\pi + A\alpha, \quad P_{yx} = \alpha\beta/4\pi + B\alpha,$$
$$P_{zx} = \alpha\gamma/4\pi + C\alpha, \text{ \&c.} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\delta).$$

Thus it is the system (α), together with the tensions (3), shears (4) and couples (5). But the result (β) shows that the system (α) *alone* will explain the forces and couples which act on a rigid magnet taken as a whole. Hence Maxwell's system (δ) is incorrect: for example it will give double as big a resultant couple as it should. This statement may be verified by determining the couple due to the action of a uniform field of magnetic force upon a sphere which is uniformly magnetised and is placed in the field with its direction of magnetisation perpendicular to the lines of force.

The error on which we wish to lay stress is that of using (in § 639) a method which suggests some but not all of the stresses which act on an element of volume: apart from this is the incorrectness of determining stresses in the ether, which fills all space, from an expression for the energy as the volume integral of a function which is zero outside the magnet.

It is found (§§ 34—37) that, if the ether is stationary and

the polarisations of electric and magnetic material media are molecular, the expression

$$\{\mathbf{E}^2 + (K-1)\,\mathbf{E}'^2 + \mathbf{H}^2 + (\mu-1)\,\mathbf{H}'^2\}/8\pi$$

as the energy per unit volume yields a system of stresses *in the ether* which agrees with Maxwell's in the case of a stationary electrostatic field. It is natural to interpret magnetism as due to the description of small orbits by the ions*, the magnetic moment of a molecule being proportional to the magnetic force acting on it. The relation of the magnetic moment per unit volume to the magnetic force will in that case (§ 47) be different from the relation of the electric moment to the electric force: and the ether-stresses in a magnetic field will be different in character from those in an electric field (§§ 50, 51). On each ion, surrounded as it is by the ether, a resultant force will be produced by the ether-stresses. And these resultant forces acting on the ions, of which the material medium is composed, will call into existence the stresses *within the bodies*; when we know the forces on the component particles of a material medium per unit volume we can find the stress within the body by the application of the ordinary laws of mechanics and of the theory of elasticity (§ 42). Thus we find (§ 52) stresses at surfaces of separation and in the interior of material media which are not Maxwellian, do not involve shears, and are consistent with the equilibrium of a fluid medium.

As Helmholtz pointed out†, Maxwell's theory labours under the disadvantage of giving a finite resultant force

$$\frac{d}{dt}\,[\mathbf{HE}]/4\pi V$$

per unit volume on a stationary element of free ether: in § 50 this difficulty does not arise.

* The extended definition of 'ion' on p. 63 would include the 'corpuscles' recently discussed by J. J. Thomson (*Nature*, May 10, 1900: *Phil. Mag.*, Feb. 1900).

† Helmholtz, *Wiss. Abhand.* III. p. 531.

In §§ 54—56 will be found a comparison of the theoretical stresses of §§ 52, 53 with observations, chiefly on liquid media, made by Quincke and others. The agreement is in all cases as close as could be expected in view of the experimental difficulties.

In Part IV. we have tried to ascertain the possibility of explaining aberration on the hypothesis that the polarisation of material media is continuous, not molecular as is assumed in Part II. §§ 19, 20.

In § 57 it is shown that the velocity of light in a medium drifting with constant velocity u through a stationary ether is increased by an amount $\frac{1}{2}u\,(K-1)/K$. In § 58 in order to obtain Fresnel's coefficient we are driven to assume that under these conditions the ether is dragged by the material medium with velocity $u\,(K-1)/(K+1)$: the ordinary facts of aberration are then explained (§§ 59, 60), for the directions of rays prove to be governed by the same laws as in the theory of Lorentz.

As might be expected, Röntgen's experiment with the spinning disc is decisive between the theories of molecular and continuous polarisation. Its explanation on the latter hypothesis is impossible if there is relative motion between the earth and the ether, and this is an inevitable part of the assumption of § 58.

We have availed ourselves of the system of Vector Algebra introduced by Heaviside and adopted by Lorentz and Föppl*: its use appears to give distinctly greater insight as well as greater brevity. A table of vector notation and formulæ is appended, as well as a list of symbols used for expressing the

* Föppl's *Einführung in die Maxwell'sche Theorie der Elektricität* contains in its first section (pp. 5—88) an extremely good introduction to vector algebra. See also Heaviside's *Electromagnetic Theory*, Vol. I. ch. III. 1893.

physical quantities involved. The units employed are those adopted by Hertz*

An attempt has been made to acknowledge indebtedness by giving references whenever assistance has been consciously received: where omissions have inadvertently been made, the largeness of the literature of the subject must be my excuse.

* Hertz, *Ges. Werke*, Vol. II. p. 213: 'Electric Waves,' p. 199.

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VECTOR NOTATION AND FORMULAE.

If the components of a vector are denoted by A_1, A_2, A_3 when the vector itself is denoted by $\mathbf{A}$ or by (A_1, A_2, A_3), we have the following scheme :—

$$\mathbf{AB} \equiv \text{the scalar product of } \mathbf{A}, \mathbf{B}$$
$$\equiv A_1B_1 + A_2B_2 + A_3B_3 = \mathbf{BA},$$
$$[\mathbf{AB}] \equiv \text{the vector product of } \mathbf{A}, \mathbf{B}$$
$$\equiv (A_2B_3 - A_3B_2,\ A_3B_1 - A_1B_3,\ A_1B_2 - A_2B_1)$$
$$= -[\mathbf{BA}] \quad \text{.....I,}$$
$$\mathbf{A}\,[\mathbf{BC}] = \mathbf{B}\,[\mathbf{CA}] = \mathbf{C}\,[\mathbf{AB}] = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \quad \text{.....II,}$$
$$[\mathbf{A}\,[\mathbf{BC}]] = \mathbf{B}\,.\,\mathbf{CA} - \mathbf{C}\,.\,\mathbf{AB} \quad \text{.....III,}$$
$$\nabla \equiv \left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right),$$
$$\text{div}\,\mathbf{A} \equiv \nabla\mathbf{A} = \frac{dA_1}{dx} + \frac{dA_2}{dy} + \frac{dA_3}{dz} \quad \text{.....IV,}$$
$$\text{curl}\,\mathbf{A} \equiv [\nabla\mathbf{A}] = \left(\frac{dA_3}{dy} - \frac{dA_2}{dz},\ \frac{dA_1}{dz} - \frac{dA_3}{dx},\ \frac{dA_2}{dx} - \frac{dA_1}{dy}\right) \quad \text{.....V,}$$
$$\mathbf{A}\nabla\,.\,\mathbf{B} = \left(A_1\frac{d}{dx} + A_2\frac{d}{dy} + A_3\frac{d}{dz}\right)\mathbf{B} \quad \text{.....VI,}$$
$$\text{div}\,[\mathbf{AB}] = \mathbf{B}\,\text{curl}\,\mathbf{A} - \mathbf{A}\,\text{curl}\,\mathbf{B} \quad \text{.....VII,}$$
$$\text{curl}\,[\mathbf{AB}] = \mathbf{A}\,\text{div}\,\mathbf{B} - \mathbf{A}\nabla\,.\,\mathbf{B} - \mathbf{B}\,\text{div}\,\mathbf{A} + \mathbf{B}\nabla\,.\,\mathbf{A} \quad \text{.....VIII,}$$
$$\nabla_B\,.\,\mathbf{AB} \equiv \left(A_1\frac{dB_1}{dx} + A_2\frac{dB_2}{dx} + A_3\frac{dB_3}{dx},\right.$$
$$A_1\frac{dB_1}{dy} + A_2\frac{dB_2}{dy} + A_3\frac{dB_3}{dy},$$
$$\left.A_1\frac{dB_1}{dz} + A_2\frac{dB_2}{dz} + A_3\frac{dB_3}{dz}\right)$$
$$= \mathbf{A}\nabla\,.\,\mathbf{B} + [\mathbf{A}\,\text{curl}\,\mathbf{B}] \quad \text{.....IX.}$$

TABLE OF NOTATION OF PHYSICAL QUANTITIES.

Vector	Components	Meaning
$\mathbf{B}$	$a,\ b,\ c$	Maxwell's 'magnetic induction' $= \mu\mathbf{H} + 4\pi\mathbf{I}$.
$\mathbf{C}$	$p,\ q,\ r$	Conduction current.
$\mathbf{D}$	$\mathfrak{X},\ \mathfrak{Y},\ \mathfrak{Z}$	$4\pi \times$ (Total electric polarisation exclusive of permanent charges) $= \mathbf{E} + \mathbf{D}'$.
$\mathbf{D}'$	$\mathfrak{X}',\ \mathfrak{Y}',\ \mathfrak{Z}'$	$4\pi \times$ (Induced electric polarisation of material medium) $= (K-1)\,\mathbf{E}'$.
$\mathbf{E}$	$X,\ Y,\ Z$	Electric force at a point fixed relative to the ether.
$\mathbf{E}'$	$X',\ Y',\ Z'$	Electric force at a point fixed relative to the moving material medium.
$\mathbf{E}''$	$X'',\ Y'',\ Z''$	Electric force at a point moving with velocity $u'',\ v'',\ w''$.
$\mathbf{F}$	$\Xi,\ \mathrm{H},\ \mathrm{Z}$	Mechanical force per unit volume.
$\mathbf{G}$	$\mathfrak{L},\ \mathfrak{M},\ \mathfrak{N}$	$4\pi \times$ (Total magnetic polarisation exclusive of permanent magnetisation) $= \mathbf{H} + \mathbf{G}'$.
$\mathbf{G}'$	$\mathfrak{L}',\ \mathfrak{M}',\ \mathfrak{N}'$	$4\pi \times$ (Induced magnetic polarisation of the material medium) $= (\mu - 1)\,\mathbf{H}'$.
$\mathbf{H}$	$L,\ M,\ N$	Magnetic force at a point fixed relative to the ether.
$\mathbf{H}'$	$L',\ M',\ N'$	Magnetic force at a point fixed relative to the moving material medium.

Vector	Components	Meaning
$\mathbf{H}''$	L'', M'', N''	Magnetic force at a point moving with velocity u'', v'', w''.
$\mathbf{I}$	A, B, C	Intensity of permanent magnetisation.
$\mathbf{M}$	$e\xi$, $e\eta$, $e\zeta$	Moment of an electric doublet.
	$m\xi$, $m\eta$, $m\zeta$	Moment of a magnetic doublet.
$\mathbf{P}$	P, Q, R	$\frac{1}{4\pi V}[\mathbf{EH}]$.
$\mathbf{P}'$	P', Q', R'	$\frac{1}{4\pi V}[\mathbf{E'H'}]$.
$\mathbf{Q}$	$\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$	$\frac{1}{4\pi V}[\mathbf{DG}]$.
$\mathbf{u}$	u, v, w	Velocity of matter relative to the free ether.

ρ', σ'	Volume and surface densities of the electricity of the induced polarisation.
ρ, σ	Volume and surface densities of the permanent electric charges.
τ', υ'	Volume and surface densities due to induced magnetisation.
τ υ	Volume and surface densities of permanent magnetisation.

CONTENTS.

PART III.

PART IV.