AXIOMATIC THEORIES OF TRUTH

At the centre of the traditional discussion of truth is the question of how truth is defined. Recent research, especially with the development of deflationist accounts of truth, has tended to take truth as an undefined primitive notion governed by axioms, while the liar paradox and cognate paradoxes pose problems for certain seemingly natural axioms for truth. In this book, Volker Halbach examines the most important axiomatizations of truth, explores their properties, and shows how the logical results impinge on the philosophical topics related to truth. For instance, he shows how the discussion of topics such as deflationism depends on the solution of the paradoxes. His book is an invaluable survey of the logical background to the philosophical discussion of truth, and will be indispensable reading for any graduate and professional philosopher in theories of truth.

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CAMBRIDGE UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Published in the United States of America by Cambridge University Press, New York

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107424425

First published 2011
Revised paperback edition 2014

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-11581-0 Hardback
ISBN 978-1-107-42442-5 Paperback

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Preface

This book has four parts. In the first part I sketch some mathematical preliminaries, fix notational conventions, and outline some motivations for studying axiomatic theories of truth. Deeper philosophical investigation, however, is postponed to the last part when the significance of the formal results is discussed. The axiomatic theories of truth and the results about them are then given in the two central parts. The first of them is devoted to typed theories, that is, to theories where the truth predicate applies provably only to sentences not containing the truth predicate. In the third part of the book I discuss type-free theories of truth and how inconsistency can be avoided without Tarski's object and metalanguage distinction. In the fourth and final part, the philosophical implications of the formal results are evaluated.

I have tried to make the book usable as a handbook of axiomatic truth theories, so that one can dip into various sections without having read all the preceding material. To this end I have also included many cross references and occasionally repeated some explanations concerning notation. It should be possible to read the final part on philosophical issues without having read the two formal parts containing the formal results. However, this last part presupposes some familiarity with the notation introduced in Chapters 5 and 6 in the first part. Of course, when discussing philosophical issues I will refer back to the formal results obtained in the two previous parts, and the reader who is interested in the last part only and skips the two formal parts will have to take my word for them.

All parts of the book should be accessible to a reader who has some acquaintance with the proofs of the Gödel incompleteness theorems and therefore with the basic concepts of recursion theory and metamathematics. In sections where I have used techniques from other areas of logic like model theory, I have defined all notions I use and have made most proofs so explicit that they should be accessible to readers not used to these techniques. I also assume very little with respect to proof theory: I do not use methods from ordinal analysis as I felt unable to provide an introduction to this branch of proof theory in a short chapter. In many research papers axiomatic truth theories are analysed by relating them to well-investigated subsystems

of second-order number theory. The truth theories can then be compared via well-known results about these subsystems. Here I have attempted, whenever possible, to relate the theories of truth to one another directly, without the detour via the second-order systems, obviating the need to appeal to well-documented or folklore results about them.

Acknowledgements. In 2009 New College granted me a term of sabbatical leave, which was extended by another term, financed by the John Fell fund. During these two terms I wrote most of this book. I am grateful for this support. I am also indebted to Eugene Ludwig for generously supporting the preparation of the final draft through the college.

I owe thanks to John Burgess, Martin Fischer, Kentaro Fujimoto, Richard Heck, and Graham Leigh for making unpublished drafts of papers available to me and allowing me to use this material in the book.

Work on the material in this book stretches back a long time and it is not feasible to list everybody who has helped with suggestions and criticism or by teaching me. I apologize to everyone I do not thank explicitly.

I am obliged to Eduardo Barrio, Andrea Cantini, Solomon Feferman, Hartry Field, Richard Heck, Richard Kaye, Jeff Ketland, Graham Leigh, Karl-Georg Niebergall, and Albert Visser for numerous discussions, comments on papers and earlier drafts of this book. I owe much to the members of the Luxemburger Zirkel Hannes Leitgeb, Philip Welch, and especially Leon Horsten. Special thanks are due to Kentaro Fujimoto and James Studd for numerous corrections and suggestions.

New College, April 2010

Note on the Paperback Edition 2014

The main difference to the original 2011 edition is in Section 8.1. Kentaro Fujimoto spotted a mistake in the proof of the conservativity of $\mathrm{CT}^{\uparrow}$ over PA, that is, Theorem 8.5 and the proof has been removed.

Ewa Bigaj, Martin Fischer, Eberhard Guhe, Harold Hodes, Carlo Nicolai and Lavinia Picollo made me aware of various typos and smaller problems, which have been corrected in the new edition. I am grateful to all who have helped me with this improved edition.

At http://users.ox.ac.uk/~sfop0114/atot.htm I maintain a web page for further updates.

New College, October 2013