## Contents

*Preface* [page ix]

Prologue: Hilbert's last problem [1]

- 1 Introduction [3]
  - 1.1 The idea of a proof [3]
  - 1.2 Proof analysis: an introductory example [4]
  - 1.3 Outline [9]

PART I PROOF SYSTEMS BASED ON NATURAL DEDUCTION

#### 2 Rules of proof: natural deduction [17]

- 2.1 Natural deduction with general elimination rules [17]
- 2.2 Normalization of derivations [23]
- 2.3 From axioms to rules of proof [29]
- 2.4 The theory of equality [32]
- 2.5 Predicate logic with equality and its word problem [35] Notes to Chapter 2 [37]

#### 3 Axiomatic systems [39]

- 3.1 Organization of an axiomatization [39]
- 3.2 Relational theories and existential axioms [46] Notes to Chapter 3 [49]
- 4 Order and lattice theory [50]
  - 4.1 Order relations [50]
  - 4.2 Lattice theory [52]
  - 4.3 The word problem for groupoids [57]
  - 4.4 Rule systems with eigenvariables [62] Notes to Chapter 4 [67]

### 5 Theories with existence axioms [68]

- 5.1 Existence in natural deduction [68]
- 5.2 Theories of equality and order again [71]
- 5.3 Relational lattice theory [73] Notes to Chapter 5 [82]

v

vi

Contents

PART II	PROOF	SYSTEMS	BASED	ON	SEQUENT
CALCULUS					

- 6 Rules of proof: sequent calculus [85]
  - 6.1 From natural deduction to sequent calculus [85]
  - 6.2 Extensions of sequent calculus [97]
  - 6.3 Predicate logic with equality [106]
  - 6.4 Herbrand's theorem for universal theories [110] Notes to Chapter 6 [111]
- 7 Linear order [113]
  - 7.1 Partial order and Szpilrajn's theorem [113]
  - 7.2 The word problem for linear order [119]
  - 7.3 Linear lattices [123]
    - Notes to Chapter 7 [128]

#### PART III PROOF SYSTEMS FOR GEOMETRIC THEORIES

- 8 Geometric theories [133]
  - 8.1 Systems of geometric rules [133]
  - 8.2 Proof theory of geometric theories [138]
  - 8.3 Barr's theorem [144] Notes to Chapter 8 [145]
- 9 Classical and intuitionistic axiomatics [147]
  - 9.1 The duality of classical and constructive notions and proofs [147]
  - 9.2 From geometric to co-geometric axioms and rules [150]
  - 9.3 Duality of dependent types and degenerate cases [155] Notes to Chapter 9 [156]
- 10 Proof analysis in elementary geometry [157]
  - 10.1 Projective geometry [157]
  - 10.2 Affine geometry [173]
  - 10.3 Examples of proof analysis in geometry [180] Notes to Chapter 10 [181]

PART IV PROOF SYSTEMS FOR NON-CLASSICAL LOGICS

- 11 Modal logic [185]
  - 11.1 The language and axioms of modal logic [185]
  - 11.2 Kripke semantics [187]
  - 11.3 Formal Kripke semantics [189]
  - 11.4 Structural properties of modal calculi [193]
  - 11.5 Decidability [201]

# CAMBRIDGE

Contents

vii

11.6 Modal calculi with equality, undefinability results [210]

11.7 Completeness [213] Notes to Chapter 11 [219]

### 12 Quantified modal logic, provability logic, & other

non-classical logics [222]

- 12.1 Adding the quantifiers [222]
- 12.2 Provability logic [234]
- 12.3 Intermediate logics [239]
- 12.4 Substructural logics [249]

Notes to Chapter 12 [251]

Bibliography[254]Index of names[262]Index of subjects[264]