Proof Analysis

This book continues from where the authors' previous book, *Structural Proof Theory*, ended. It presents an extension of the methods of analysis of proofs in pure logic to elementary axiomatic systems and to what is known as philosophical logic. A self-contained brief introduction to the proof theory of pure logic that serves both the mathematically and philosophically oriented reader is included. The method is built up gradually, with examples drawn from theories of order, lattice theory and elementary geometry. The aim is, in each of the examples, to help the reader grasp the combinatorial behaviour of an axiom system, which typically leads to decidability results. The last part presents, as an application and extension of all that precedes it, a proof-theoretical approach to the Kripke semantics of modal and related logics, with a great number of new results, providing essential reading for mathematical and philosophical logicians.

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Proof Analysis

A Contribution to Hilbert's Last Problem

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CAMBRIDGE
UNIVERSITY PRESS

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University Printing House, Cambridge CB2 8BS, United Kingdom

Published in the United States of America by Cambridge University Press, New York

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107417236

First published 2011
First paperback edition 2014

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data
Negri, Sara, 1967–
Proof analysis : a contribution to Hilbert's last problem / Sara Negri, Jan von Plato.
p. cm.
Includes bibliographical references and index.
ISBN 978-1-107-00895-3 (hardback)
1. Proof theory. I. Von Plato, Jan. II. Title.
QA9.54.N438 2011
511.3′6 – dc23 2011023026

ISBN 978-1-107-00895-3 Hardback
ISBN 978-1-107-41723-6 Paperback

Cambridge University Press
978-1-107-41723-6 - Proof Analysis: A Contribution to Hilbert's Last Problem
Sara Negri and Jan Von Plato
Frontmatter
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Preface

Proof theory, one of the two main directions of logic, has been mostly concentrated on pure logic. There have been systematic reasons to think that such a limitation of proof theory to pure logic is inevitable, but about twelve years ago, we found what appears to be a very natural way of extending the proof theory of pure logic to cover also axiomatic theories. How this happens, and how extensive of our method is, is explained in this book. We have written it so that, in principle, no preliminary knowledge of proof theory or even of logic is necessary.

The book can be profitably read by students and researchers in philosophy, mathematics, and computer science. The emphasis is on the presentation of a method, divided into four parts of increasing difficulty and illustrated by many examples. No intricate constructions or specialized techniques appear in these; all methods of proof analysis for axiomatic theories are developed by analogy to methods familiar from pure logic, such as normal forms, subformula properties, and rules of proof that support root-first proof search. The book can be used as a basis for a second course in logic, with emphasis on proof systems and their applications, and with the basics of natural deduction and sequent calculus for pure logic covered in Part I, Chapter 2, and Part II, Chapter 6.

A philosopher who seeks the general significance of the work should be able to see in what sense it contributes to the solution of a fascinating recently discovered *last problem of Hilbert* that belongs to proof theory. The much later *Hilbert programme* had more specific aims. It is remarkable how many of the original aims of this foundational programme can be carried through in, say, algebra and geometry, and indeed in many parts of mathematics that do not involve the natural numbers and the incompleteness of their theory.

Mathematically oriented readers should be able, after a study of this book, to produce independent work on the application of the method of proof analysis in their favourite axiomatic theories.

The fourth part, on non-classical logics, is mainly aimed at the student and specialist in philosophical logic. It presents in a systematic form, building on the previous parts, a proof theory of non-classical logics, with semantical aspects incorporated through what are known as labelled logical calculi. The fundamental idea here is very clear: the various systems of modal and other non-classical logics can usually be characterized by some key properties, expressed as conditions in the standard Kripke semantics. These conditions are, taken abstractly, axioms for the frames of the said semantics, and they convert into rules that extend an underlying sequent calculus. The execution of this idea in the fourth part builds, by way of the method, on virtually everything that has been presented in the previous parts. It was a great surprise to the authors when the first of them discovered the application of proof analysis to non-classical logics in 2003, and many results in Part IV are new. This part is also useful for the study of logic in computer science. Recent years have seen a growth of literature in computer science on logical systems of knowledge presentation that stems from epistemic logic as developed by philosophers, and to which systems the method of Part IV can be fruitfully applied.

Hilbert's enigmatic last problem that decorates our title is explained in our Prologue that begins the book. The structure of the book is explained later in Section 1.3, after which a summary of the individual chapters follows. Finally, a word about what is not included: we have decided to, by and large, present our approach and let it speak for itself. Of the different parts of proof theory, we have a lot to say about structural proof theory, the topic of our previous book published in 2001. Other topics, such as the proof theory of arithmetic, ordinal proof theory, and what Anne Troelstra calls interpretational proof theory in his *Basic Proof Theory*, remain largely untouched. Troelstra's book can be consulted for a first look at these different aspects to proof theory. There is no easy introduction to the proof theory of arithmetic, but Takeuti's *Proof Theory*, especially in its early chapters, is fairly accessible. The recent book by Pohlers (2009) on ordinal proof theory is a hard read. Kohlenbach's (2008) hefty tome collects together an enormous amount of results that belong to interpretational proof theory.

This book began with a series of lectures titled 'Five Lectures on Proof-Analysis' that the first author gave in Dresden in 2003. A second series was given in Munich, and a third in Braga, Portugal, in 2006. The year after, the second author gave a more extensive course on the topic at the University of Helsinki. We thank those involved, organizers, colleagues, and students, for these opportunities. In particular, we thank Roy Dyckhoff as well as our students Bianca Boretti, Annika Kanckos, and Andrea Meinander, who have all done research that has affected our presentation. We have also benefited from comments by Michael von Boguslawski and Sergei Soloviev. All the

while, we have been surrounded by the patient wondering of Daniel, Stella, and Niclas.

In 1998, when the program of proof analysis was launched by our joint article 'Cut Elimination in the Presence of Axioms' in *The Bulletin of Symbolic Logic*, we received from Jussi Ketonen the following reaction: 'I suspect that this type of work will eventually lead to a completely new kind of understanding of proofs – not only as applications of rules, axioms, or ideologies, but as a branch of mathematics'. Now, twelve years later, we hope to have realized at least a beginning of that vision.