Part I

Theoretical framework

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Field theory review

Quantum field theory is the language in terms of which the laws of physics are cast, and so we start with a whirlwind summary of some of its main features. Interspersed amongst the introductory topics in this chapter we also discuss some of the more general features that are usually demanded of any reasonable field-theoretic description of nature.

1.1 Hilbert space, creation and annihilation operators

Quantum field theories are special kinds of quantum mechanical theories which describe the behavior of particles. As quantum mechanical theories, their most basic objects are the Hilbert space of possible states $\mathcal{H}$, and the Hamiltonian H which describes time evolution in that Hilbert space.

The possible kinds of states are zero-particle states, one-particle states, two-particle states, and so on. Therefore, the Hilbert space in which all operators live is the tensor product of the zero-particle space with the one-particle space with the two-particle space, and so on:

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \tag{1.1}$$

Here

$$\mathcal{H}_0 = \{|0\rangle\} \tag{1.2}$$

denotes the one-dimensional space spanned by the zero-particle state or vacuum: $|0\rangle$.

$$\mathcal{H}_1 = \{|\mathbf{p}, k\rangle\} \tag{1.3}$$

is similarly the span of all one-particle states with the basis states chosen to be eigenstates of linear momentum. Here $\mathbf{p}$ represents the momentum of a state, and k denotes all of the other particle labels.

The space of N-particle states is constructed as the tensor product of N copies of the one-particle space. For instance, $\mathcal{H}_2$ is the set of all two-particle states,

$$\mathcal{H}_2 = \{|\mathbf{p}_1, k_1; \mathbf{p}_2, k_2\rangle = \pm|\mathbf{p}_2, k_2; \mathbf{p}_1, k_1\rangle\} \tag{1.4}$$

etc. The sign, $\pm$, is $+$ for bosons and $-$ for fermions. A Hilbert space constructed in this way is conventionally referred to as a Fock space.

It is convenient to express the operators that act within this space in terms of a basic set of *creation* and *annihilation* operators in the following way. The *annihilation operator*, $a_{\mathbf{p}k}$, is the operator that removes the particle with quantum numbers $\mathbf{p}$ and k from a given state. If the state on which $a_{\mathbf{p}k}$ acts does not contain the particle in question then the operator is defined to give zero. That is,

$$\begin{aligned} a_{\mathbf{p}k}|0\rangle &= 0 \\ a_{\mathbf{p}k}|\mathbf{q}, l\rangle &= 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta_{kl}|0\rangle \\ a_{\mathbf{p}k}|\mathbf{q}, l; \mathbf{k}, m\rangle &= 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta_{kl}|\mathbf{k}, m\rangle \\ &\quad \pm 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{k})\delta_{km}|\mathbf{q}, l\rangle \end{aligned} \tag{1.5}$$

and so on. Here, $E_{\mathbf{p}}$ is the energy of a particle of spatial momentum $\mathbf{p}$, namely, $\sqrt{\mathbf{p}^2 + m^2}$, with m the mass of a particle with labels k. The sign in this last result is $\pm$ according to the statistics of particles $|\mathbf{p}, k\rangle$ and $|\mathbf{q}, l\rangle$. Here and throughout, we use units for which $\hbar = c = 1$. The normalization is chosen to make Lorentz invariance more manifest, as discussed below.

This definition implies that the Hermitian conjugate, $a^*_{\mathbf{p}k}$, of $a_{\mathbf{p}k}$ is a *creation operator* for the same particle type; i.e.

$$a^*_{\mathbf{p}i}|0\rangle = |\mathbf{p}, i\rangle \tag{1.6}$$

$$a^*_{\mathbf{p}i}|\mathbf{q}, j\rangle = |\mathbf{p}, i; \mathbf{q}, j\rangle \tag{1.7}$$

etc. (Our notation is to use an asterisk for complex conjugation of c-numbers and Hermitian conjugation of operators, and to reserve a dagger, †, for Hermitian conjugation of matrices.)

These definitions, together with the normalization convention

$$\langle \mathbf{p}, i|\mathbf{q}, j\rangle = 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta_{ij} \tag{1.8}$$

imply the following properties. For bosons,

$$|\mathbf{p}, i; \mathbf{q}, j\rangle = |\mathbf{q}, j; \mathbf{p}, i\rangle \tag{1.9}$$

$$[a_{\mathbf{p}i}, a_{\mathbf{q}j}] = \left[a^*_{\mathbf{p}i}, a^*_{\mathbf{q}j}\right] = 0 \tag{1.10}$$

$$\left[a_{\mathbf{p}i}, a^*_{\mathbf{q}j}\right] = 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta_{ij} \tag{1.11}$$

and for fermions,

$$|\mathbf{p},i;\mathbf{q},j\rangle = -|\mathbf{q},j;\mathbf{p},i\rangle \tag{1.12}$$
$$\{a_{\mathbf{p}i}, a_{\mathbf{q}j}\} = \{a^*_{\mathbf{p}i}, a^*_{\mathbf{q}j}\} = 0 \tag{1.13}$$
$$\{a_{\mathbf{p}i}, a^*_{\mathbf{q}j}\} = 2E_{\mathbf{p}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})\delta_{ij} \tag{1.14}$$

in which $[A,B] = AB - BA$ and $\{A,B\} = AB + BA$.

A few comments are in order about the field normalizations above. First, note that momentum integrations $\mathrm{d}p/2\pi$ always have factors of 2π in the denominator, and momentum delta functions $2\pi\delta(p-q)$ always have factors of 2π multiplying them. Following these rules,

- momentum space and energy integrations always involve $\int \mathrm{d}^3\mathbf{p}/(2\pi)^3$, $\int \mathrm{d}E/2\pi$;
- delta functions are always of form $(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})$ or $(2\pi)\delta(E_1-E_2)$,

accounts for all 2π factors we will ever encounter.

Second, the momentum delta functions we have written are accompanied by factors of $2E_{\mathbf{p}}$, and the same $2E_{\mathbf{p}}$ appears in the denominator in momentum integrations. This normalization, called relativistic normalization, is convenient in a Lorentz invariant theory, because it makes it easier to make Lorentz invariance manifest. Note in particular, that

$$\int \frac{\mathrm{d}^3\mathbf{p}}{2E_{\mathbf{p}}(2\pi)^3} = \int \frac{\mathrm{d}^4p}{(2\pi)^4} 2\pi\delta(p^2+m^2)\theta(p^0) \tag{1.15}$$

which is manifestly Lorentz invariant. [Note that our metric convention is that $\eta_{\mu\nu} = \mathrm{Diag}[-1,1,1,1]$, so $p^2 \equiv \eta_{\mu\nu}p^\mu p^\nu = -(p^0)^2 + \mathbf{p}^2$.] This expression can be verified by performing the p^0 integration, using the δ function. Its Lorentz invariance is not quite manifest, since the step function $\theta(p^0)$ does not look invariant, as it refers to the time component; but the $2\pi\delta(p^2+m^2)$ forces p^μ to be timelike for $m^2 > 0$ and lightlike for $m^2 = 0$, which ensures that the sign of p^0 does not change under (orthochronous) Lorentz transformations. Throughout this book, whenever there is an integral $\int \mathrm{d}^3p/(2\pi)^3 2E_{\mathbf{p}}$, we will always implicitly define $p^0 = E_{\mathbf{p}}$ inside the integral.

The fundamental claim now to be made is that *any* operator acting on our Hilbert space, $\mathcal{H}$, can be written as a linear combination of monomials of the as and a^*s; i.e.,

$$\mathcal{O} = A_{0,0} + \sum_i \int \frac{\mathrm{d}^3p}{2E_{\mathbf{p}}(2\pi)^3}\left[A_{0,1}(\mathbf{p},i)a_{\mathbf{p}i} + A_{1,0}(\mathbf{p},i)a^*_{\mathbf{p}i}\right] \tag{1.16}$$

$$+\sum_{ij}\int \frac{\mathrm{d}^3p\,\mathrm{d}^3q}{4E_{\mathbf{p}}E_{\mathbf{q}}(2\pi)^6}\Big[A_{0,2}(\mathbf{p},i;\mathbf{q},j)a_{\mathbf{p}i}a_{\mathbf{q}j}+A_{1,1}(\mathbf{p},i;\mathbf{q},j)a^*_{\mathbf{p}i}a_{\mathbf{q}j}$$
$$+A_{2,0}(\mathbf{p},i;\mathbf{q},j)a^*_{\mathbf{p}i}a^*_{\mathbf{q}j}\Big]+\cdots \tag{1.17}$$

The operators, $\mathcal{O}$, are in one-to-one correspondence with the coefficient functions $\{A_{0,0}, A_{1,0}(\mathbf{p},i), A_{0,1}(\mathbf{p},i),\ldots\}$. This can be shown inductively by explicitly solving for these coefficients in terms of the matrix elements of $\mathcal{O}$: $\langle\psi|\mathcal{O}|\phi\rangle$. For example $\langle 0|\mathcal{O}|0\rangle = A_{0,0}$, $\langle 0|\mathcal{O}|\mathbf{p},i\rangle = A_{0,1}(\mathbf{p},i)$, and so on.

In particular, the Hamiltonian for a system of free particles has a simple expression in terms of the as and a^*s:

$$H_0 = E_0 + \sum_i \int \frac{\mathrm{d}^3p}{2E_{\mathbf{p}}(2\pi)^3}\,\varepsilon(\mathbf{p},i)a^*_{\mathbf{p}i}a_{\mathbf{p}i}. \tag{1.18}$$

To learn the interpretation of the coefficients E_0 and $\varepsilon(\mathbf{p},i)$, calculate the action of H_0 on various states. On the vacuum H_0 gives

$$H_0|0\rangle = E_0|0\rangle \tag{1.19}$$

since $a_{\mathbf{p}i}|0\rangle = 0$. E_0 is clearly the energy of the no-particle state $|0\rangle$, i.e. the vacuum energy. Similarly,

$$H_0|\mathbf{q},j\rangle = [E_0 + \varepsilon(\mathbf{q},j)]\,|\mathbf{q},j\rangle \tag{1.20}$$

and

$$H_0|\mathbf{q}_1,j_1;\ldots;\mathbf{q}_N,j_N\rangle = \left[E_0 + \sum_{k=1}^{N}\varepsilon(\mathbf{q}_k,j_k)\right]|\mathbf{q}_1,j_1;\ldots;\mathbf{q}_N,j_N\rangle \tag{1.21}$$

etc. The many-particle momentum eigenstates, $|\mathbf{q}_1,j_1;\ldots;\mathbf{q}_N,j_N\rangle$ are also eigenstates of the energy, H_0, with eigenvalue

$$E = E_0 + \sum_{k=1}^{N}\varepsilon(\mathbf{q}_k,j_k). \tag{1.22}$$

This implies that the energy of a single-particle state $|\mathbf{p},i\rangle$ relative to the vacuum is $\varepsilon(\mathbf{p},i)$. Relativistic kinematics then determines the momentum-dependence of ε on $\mathbf{p}$ as

$$\varepsilon(\mathbf{p},i) = \sqrt{\mathbf{p}^2 + m_i^2} = E_{\mathbf{p}} \tag{1.23}$$

where m_i is the mass of particle type i. Notice that the energy of a many-particle state relative to the vacuum is just the sum of the single-particle energies, showing that the particles described by H_0 do not interact.

We emphasize that this is a special property of free field theories; in

general, even if single-particle states are eigenstates of the Hamiltonian, many-particle states are in general not eigenstates of the Hamiltonian. This means that they can undergo non-trivial time evolution. Indeed, almost all interesting phenomena in particle physics arise from the fact that many-particle states are not eigenstates of the Hamiltonian.

1.2 General properties of interactions

We are interested in writing down a Hamiltonian

$$H = H_0 + H_{\rm int} \tag{1.24}$$

that describes the interactions of the particles we know. The present section is devoted to summarizing the minimal requirements for a physically reasonable theory. These properties translate into a set of restrictions on what form will be allowed for H. The purpose of this process is to arrive at the general class of theories from which the standard model is to be chosen. Being aware of the alternatives available gives some feeling for which features may be changed and which are inviolable.

We now return to a statement of these requirements. A sketch of their justification is given in the next subsection, but for a complete discussion the reader should consult a field theory text.

1.2.1 Physical constraints on H

The basic principles we demand of any candidate physical theory are:

(i) Unitarity: (i.e. conservation of probability)
The requirement here is to ensure that time evolution preserve the property that the sum of probabilities over all mutually exclusive events gives one. This requires that the time-evolution operator

$$U = \mathrm{e}^{-\mathrm{i}Ht} \tag{1.25}$$

be unitary. Equivalently the Hamiltonian must be Hermitian:

$$H = H^*. \tag{1.26}$$

(ii) Cluster decomposition: (i.e. locality)
This requirement is that physics be independent at different points in space at a given time. Specifically we require that amplitudes (and hence probabilities) for events that are well separated from one another factorize into a product of independent amplitudes. Such a

factorization is what would be expected for statistically independent events.

The condition that physics at spatially separated positions be independent comes in two parts. The first is that physical observables must commute at spatially separated points and the second is that time evolution must preserve this property. We consider each of these in turn:

(a) Microcausality
The first condition is to require that physical observables may be separately measurable at different positions and equal times. In a quantum theory we must therefore demand that all physical observables commute at space-like separations. That is:

$$[A(x), B(y)] = 0 \quad \text{for} \quad (x-y)^2 > 0. \tag{1.27}$$

Condition (1.27) is sometimes referred to as the requirement of microcausality.

(b) Locality
We next require that this property, that spatially separated physical amplitudes must factorize, be preserved by time evolution, provided, of course, that no physical signals propagate from one point to the other. Since the time-evolution operator, Eq. (1.25), is the exponential of the Hamiltonian, the property that it factorizes turns out to require that the Hamiltonian should be the sum of those for each of the spatially separated regions. The Hamiltonian must therefore have the form

$$H = \int \mathrm{d}^3x \; \mathcal{H}(\mathbf{x}, t) \tag{1.28}$$

which boils down to requiring that the total energy be a sum of the energy of the degrees of freedom at each point. This is consistent with the intuition that the degrees of freedom at each point of space at a given time are independent, since the total energy for a set of independent systems is the sum of the energies of the independent constituents.

(iii) Invariance under Lorentz transformations and translations (Poincaré invariance)
Here we build in the requirements of special relativity and translation invariance in space and time. In quantum mechanics this implies the existence of corresponding conserved charges, P^μ and $J^{\mu\nu} = -J^{\nu\mu}$

(with $\mu, \nu = 0, 1, 2, 3$), representing four-momentum and angular momentum respectively. In particular, the total energy is given by

$$H = P^0$$

The particle states transform under unitary representations of the Poincaré group given by the operators:

$$U(a, \omega) = \exp\left[-\mathrm{i} a_\mu P^\mu + \frac{\mathrm{i}}{2}\omega_{\mu\nu} J^{\mu\nu}\right] \tag{1.29}$$

generated by these conserved charges. The states, $|\mathbf{p}, \sigma, j\rangle$, may then be labelled by their three-momenta, $\mathbf{p}$, mass, m, total spin, s, and spin-projection, σ, together with any other internal labels, j. The labels m and s are generally not explicitly indicated.

The Minkowski-space conventions used in what follows are:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{1.30}$$

$$P^\mu = (E, \mathbf{p}), \quad \text{and} \quad P_\mu = (-E, \mathbf{p}) \tag{1.31}$$

$$x^\mu = (t, \mathbf{x}), \quad \text{and} \quad x_\mu = (-t, \mathbf{x}) \tag{1.32}$$

$$\epsilon^{0123} = +1 \tag{1.33}$$

implying that the invariant product $x^2 = -(x^0)^2 + \mathbf{x}^2$ is negative for timelike vectors and positive for space-like vectors. We provide a review of Lorentz symmetry in Appendix C.

(iv) Stability:
The final condition to be imposed is that the spectrum of H be bounded from below. This is necessary if the vacuum state, defined as the state of lowest energy, is to exist.

1.2.2 Renormalizability

A further condition to be imposed on the standard model that is not as fundamental as those just described is the requirement of renormalizability. In fact, perfectly good theories, such as general relativity, are not renormalizable and yet are still very successful at accounting for experiments. Some explanation is therefore required to justify this demand.

The physical motivation comes from the idea that physical theories generically come with an implicit minimum distance, d, (or maximum energy, Λ)

beyond which they are not meant to apply. For example, the quantum electrodynamics of electrons and photons is only physically correct up to an energy of twice the mass of the lightest particle that is heavier than the electron: $\Lambda = 2m_\mu$, i.e. twice the muon mass. At energies higher than this, muons can no longer be neglected, since they can be pair-produced in the final state even if they are not present initially. The correct theory for physics at energies above Λ becomes the quantum electrodynamics of photons, electrons and muons. This theory is in turn only valid up to the next threshold, the pion mass, and so on.

Classically, it is not important to specify this "cut-off" carefully. In a quantum theory, however, since all states can contribute to any given process as intermediate (or "virtual") particles, any quantum calculation will depend explicitly on the cut-off scale, Λ. This may be seen, for example, by considering the expression, in time-independent perturbation theory, for the quadratic energy shift due to a perturbing Hamiltonian,

$$\delta E_\psi = \sum_n{}' \frac{|\langle\psi|H|n\rangle|^2}{E_\psi - E_n} \qquad |n\rangle \neq |\psi\rangle \tag{1.34}$$

Clearly any state, $|n\rangle$, contributes to Eq. (1.34) regardless of its energy. Given our ignorance of the spectrum above the energy Λ, it only makes sense to include those states with energy less than Λ in this sum. The result therefore depends explicitly on Λ in a potentially complicated way.

If detailed knowledge of physics at the Λ scale is necessary in order to calculate probability amplitudes for processes at energies lower than Λ, then the theory is called *non-renormalizable*. These theories have less predictive power, since predictions depend on physics at the scale Λ, about which we are by assumption quite ignorant.

In *renormalizable* theories, on the other hand, Λ only appears in physical predictions (for large Λ) through a small number of parameters, such as the masses and charges of some or all of the particles involved. All other processes may then be computed in terms of these parameters. Once the few incalculable parameters are taken from experiment, definite predictions may be made.

Whether or not a renormalizable theory should be expected to describe a given system depends therefore on the properties of the system. Physically, successful description in terms of a renormalizable theory is equivalent to the statement that the physics of interest, at energies $E \ll \Lambda$, is largely insensitive to the higher-energy physics appropriate to the scale Λ. In general, a renormalizable description of the physics at an energy E is justified